Phys 535 **Exercises**

Deadline: Friday 7 October 2005

Question 1 *Eigenfrequencies of a string*

Consider a nonuniform string of length 1 under a tension force $F = 1$. The mass load is $\eta(x) \equiv dm/dx = 0.05 + 1.9\,x$.

Find the first four eigenfrequencies and plot the corresponding eigenfunctions. [Hint: What are good guesses to start the iteration?]

Compare the eigenfrequencies with the case $\eta = 1$.

Question 2 *Internal structure of a fully convective star*

(a) In a fully convective star, entropy

$$s = c_v(\ln p - \gamma \ln \varrho)$$

is (practically) constant. Here p and ϱ denote gas pressure and density, while c_v and c_p are the specific heats at constant density and pressure, respectively, and $\gamma \equiv c_p/c_v$.

Assuming that c_v and c_p are constant, find a relation to express pressure p in terms of density ϱ.

(b) Using that relation and the hydrostatic equation

$$\frac{dp}{dr} = -g\varrho$$

(where $g = -g_r$ is the gravitational acceleration, counted positive if inwards), show that density satisfies the equation

$$\frac{d\varrho}{dr} = -C_1 g \varrho^{2-\gamma} \; ,$$

and thus, using $g = Gm/r^2$,

$$\frac{d\varrho}{dr} = -C_2 \frac{m\varrho^{2-\gamma}}{r^2} \; , \qquad (1)$$

Relate the constant C_2 to entropy.

(c) Show that the mass function $m(r)$ (the mass enclosed inside a sphere of radius r) satisfies

$$\frac{dm}{dr} = 4\pi r^2 \varrho \; . \qquad (2)$$

(d) Using Equations (1) and (2), together with formulate a system of differential equations for $\varrho(r)$ and $m(r)$ and discuss the boundary conditions

$$\varrho(0) = \varrho_c , \tag{3}$$
$$m(0) = 0 , \tag{4}$$
$$\varrho(R) = 0 , \tag{5}$$
$$m(R) = M , \tag{6}$$

where R and M are the radius and total mass of the star, and ϱ_c is the central density. Hint: Equation (5) is only approximately valid. It may not look very natural, but remember how ϱ relates to p.

(e) If we are given the central density $\varrho_c = \varrho(0)$ and the total mass M of the star, Equations (1)–(5), together with the equation

$$\frac{dC_2}{dr} = 0 \qquad (C_2 \text{ is just an unknown constant}) \tag{7}$$

represent a system of three ODEs with four boundary conditions and a free boundary (we do not know R). Transform it into a standard boundary value problem.

(f) Solve that boundary value problem numerically for the values $\gamma = 5/3$, $M = 0.21\,\mathrm{M}_\odot = 4.2 \times 10^{29}\,\mathrm{kg}$, and $\varrho_c = 15.5\,\mathrm{M}_\odot/R_\odot^3$.

Hints:

(a) Use the (somewhat) natural units $M_\odot$ and $R_\odot$ (solar mass and radius);

(b) one of the equations is singular at $r = 0$; to overcome this, start the integration at 10^{-6} instead;

(c) use $C_2 = 1$ and $R = 0.5$ to start the shooting;

(d) you will need to truncate ϱ to ensure it does not become negative (just setting it to zero where it is negative should be enough);

(e) if you get the error message 'Singular Jacobian in broyden', it is likely that either `pde()` or the function you supply to `broyden()` is wrong.

(g) Plot $m(r)$, $g(r)$, $\varrho(r)$, and $p(r)$. Plot ϱ and p semi-logarithmically and use a reasonable `YRANGE`. Arrange the plots as four subplots in one panel. Hint: you can use `psa, /LANDSCAPE` for printing.