Evolving Magnetic Fields and the Conservation of Magnetic Moment

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3 November 1997

Abstract

It is shown that the magnetic moment $\vec{\mu}$ is a conserved quantity not only in MHD, but also in general electrodynamics under certain not very restrictive conditions. The propagation of magnetic moment from a region $\mathcal{D}$ with an evolving current system (e.g. due to dynamo action) is discussed for the two cases of vacuum and a conducting medium, respectively, surrounding $\mathcal{D}$. In the case of vacuum, the MHD approximation no longer holds and the weak electromagnetic wave emitted from $\mathcal{D}$ is important, as was pointed out by Sokoloff (1997). In the case of an unbounded conducting medium, the classical definition of $\vec{\mu}$ is generalised and $\vec{\mu}$ is shown to propagate diffusively, undisturbed by the newly generated magnetic field.

KEY WORDS: *magnetic moment, cosmic dynamos, electrodynamics*

1 Introduction

The magnetic moment $\vec{\mu}$ can be defined as the principal-value integral of the magnetic flux density over the whole space (cf. Moffatt, 1978)

$$\vec{\mu} \;:=\; \frac{3}{2\mu_0}\ \mathrm{v.\,p.}\!\int \mathbf{B}(\mathbf{x})\,dx^3 \tag{1}$$

$$\;=\; \frac{1}{2}\int \mathbf{x}{\times}\mathbf{j}(\mathbf{x})\,dx^3\ , \tag{2}$$

where

$$\mathrm{v.\,p.}\!\int \mathbf{B}\,dx^3 := \lim_{R\to\infty}\int\limits_{\mathcal{K}_R} \mathbf{B}\,dx^3 \tag{3}$$

denotes the three-dimensional principal-value integral, and $\mathcal{K}_R{:=}\{\mathbf{x}|\ \mathbf{x}^2{<}R^2\}$ is a solid sphere of radius R centred at the origin. The transition from Equation (1) to (2) is best carried out by applying the Biot-Savart law, cf. also Section 4. The second integral, (2), gives the standard definition of the magnetic moment of a current system filling a bounded domain. Unlike (1), it does not exist for a dynamo embedded into a medium of homogeneous conductivity $\sigma{\neq}0$, where $\mathbf{j}(\mathbf{x}){=}\mathcal{O}(1/r^3)$ (cf. Dobler &

Rädler, 1997). It could also be extended to a principal value, however it is more suitable to apply the generalisation given in Section 4. In situations where displacement currents are important, the current field $\mathbf{j}$ in Equation (2) is the combination of conduction current $\mathbf{j}_{\text{cond}}$ caused by the motion of charges, and displacement current $\dot{\mathbf{D}}$, i. e. $\mathbf{j} = \mathbf{j}_{\text{cond}} + \dot{\mathbf{D}}$. We will return to this point in Section 3.

Arnold *et al* (1982) pointed out that in the case of (periodic, as well as infinite) conducting space surrounding a dynamo, the integral (1) is a conserved quantity, as can be seen by integrating the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{u}\times\mathbf{B} - \eta \operatorname{curl}\mathbf{B}) \tag{4}$$

over $\mathbb{R}^3$ (in the principal-value sense) or over a cell of periodic space. In the periodic case, $\vec{\mu}$ is a topological invariant. Their argument is not as general as our result in Section 2 since it is based on the induction equation, which is derived by neglecting displacement currents. However, displacement currents do not play a rôle in conducting media filling the whole space, cf. Section 4.

In a recent paper, Sokoloff (1997) drew attention on the fact that this conservation of magnetic moment apparently contradicts the possibility of growing (or, generally, evolving) magnetic fields due to dynamo action. To solve this dilemma he showed that due to the contribution $\vec{\mu}_{\text{ext}}$ from outside the dynamo region $\mathcal{D}$, the internal magnetic moment $\vec{\mu}_{\text{int}} := \int_{\mathcal{D}} \mathbf{B}(\mathbf{x})\, dx^3$ can change with time as $\vec{\mu}_{\text{ext}}$ compensates for this variation, keeping the *total* magnetic moment $\vec{\mu} = \vec{\mu}_{\text{int}} + \vec{\mu}_{\text{ext}}$ constant.

In the case of a dynamo embedded into vacuum, the induction equation (4) describes the physics of only the conducting region. Outside, the full Maxwell equations have to be considered, and thus the magnetic moment seems not necessarily to be conserved.

The aim of the present paper is to show that conservation of magnetic moment is given in a much more general context than that of (quasi-stationary) magnetohydrodynamics and to illustrate the transport of magnetic moment away from a dynamo region in the two cases of surrounding vacuum, and homogeneously conducting plasma.

2 Conservation of magnetic moment in general electrodynamics

Maxwell's equations

$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5}$$

$$\operatorname{div}\mathbf{E} = \frac{\varrho_{\text{el}}}{\varepsilon} \tag{6}$$

allow us to represent the electric field in the following way:

$$\mathbf{E}(\mathbf{x}) = \underbrace{\frac{1}{4\pi\varepsilon}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\varrho_{\text{el}}(\mathbf{x}')\, dx'^3}_{=:\mathbf{E}_{\text{pot}}} + \underbrace{\frac{1}{4\pi}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \times \dot{\mathbf{B}}(\mathbf{x})\, dx'^3}_{=:\mathbf{E}_{\text{sol}}} \tag{7}$$

(with $\dot{\mathbf{B}}{:=}\partial\mathbf{B}/\partial t$), provided that the fields $\mathbf{E}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$ and $\varrho_{\rm el}(\mathbf{x})$ are square integrable. The square integrability of electric and magnetic field is a natural assumption in dynamo theory, since dynamos are usually localised systems, generating fields of finite energy. Since $\operatorname{curl}\mathbf{E}_{\rm pot}{\equiv}\mathbf{0}$, integration of Equation (5) over the sphere $\mathcal{K}_R$ yields

$$\frac{d}{dt}\int\limits_{\mathcal{K}_R}\mathbf{B}\,dx^3=\int\limits_{\partial\mathcal{K}_R}\mathbf{E}_{\rm sol}\times d\mathbf{f}\ . \tag{8}$$

Let us now suppose that at time $t{=}t_0$ the solenoidal part of the electric field decays like

$$\mathbf{E}_{\rm sol}(\mathbf{x})=\mathcal{O}\left(\frac{1}{r^{2+\varepsilon}}\right)\quad\text{for } r{=}|\mathbf{x}|{\to}\infty \tag{9}$$

with $\varepsilon{>}0$. Then, due to the finite speed of propagation inherent to Maxwell's equations, relation (9) holds for all times $t{>}t_0$ as well, and in the limit $R{\to}\infty$ we immediately get[1]

$$\boxed{\frac{d}{dt}\ \mathrm{v.\,p.}\!\int\mathbf{B}\,dx^3=\mathbf{0}}\ . \tag{10}$$

This means that the magnetic moment $\vec{\mu}$ is conserved not only in the framework of magnetohydrodynamics, but also, under the assumptions just made, in general electrodynamics.

Our assumption (9) is not as restrictive as it might seem. One field of realisation are models that start with a strictly localised magnetic field, that is,

$$\mathbf{B}(\mathbf{x})=\mathbf{0}\quad\text{for } |\mathbf{x}|{>}K\ , \tag{11}$$

with some constant K. Almost any dynamo system can be approximated arbitrarily good in this way, if only the value of K is chosen large enough. Another class of models compatible with condition (9) are dynamos with exponential time behaviour

$$\mathbf{B}\sim\exp(\gamma t) \tag{12}$$

embedded into vacuum (case *I*), or into a medium of constant conductivity σ (case *II*).

In case *I*, Maxwell's equations together with the ansatz (12) yield Helmholtz equations for $\mathbf{E}$ and $\mathbf{B}$, and one gets

$$\dot{\mathbf{B}}\sim\frac{e^{-\kappa r}}{r}\qquad\text{for } r{\to}\infty \tag{13}$$

with $\kappa{=}\pm\gamma/c$, the sign chosen such that $\mathfrak{Re}\,\kappa{\geq}0$. If γ has a non-vanishing real part, $\dot{\mathbf{B}}$ and hence $\mathbf{E}_{\rm sol}$ decay exponentially with distance r and thus (9) is fulfilled. For $\gamma{=}i\omega$, $\omega{\in}\mathbb{R}$, we have an oscillating dynamo emitting a monochromatic electromagnetic wave

$$\mathbf{B},\mathbf{E}\sim\frac{e^{i\omega(t-r/c)}}{r} \tag{14}$$

into the surrounding space. Since the fields (14) are obviously not square integrable, this case is relevant to dynamo theory only as the limit of cases with finite magnetic energy, all of which can be chosen to fulfil our condition (9).

[1] We write total time derivatives, since the *global* quantity $\vec{\mu}=\ \mathrm{v.\,p.}\!\int\mathbf{B}\,dx^3$ is a function of t only.

In case *II*, one also gets a Helmholtz-type equation and hence Equation (13), with $\kappa{=}\sqrt{\gamma\mu_0\sigma}$ (cf. Dobler & Rädler, 1997), where the complex square root is chosen such that $\mathfrak{Re}\,\sqrt{z}\geq 0\ \forall z$. For $\gamma{\notin}\mathbb{R}_0^-$, we thus again have exponential decay for $r{\to}\infty$, complying with (9). For real and negative γ, most of the modes are again not square integrable, cf. Dobler & Rädler (1997).

We note without proof that the Helmholtz equations mentioned yield $\vec{\mu}{=}\mathbf{0}$ in both, case *I* (with $\mathfrak{Re}\,\gamma\neq 0$) and case *II* (with $\gamma\notin\mathbb{R}_0^-$). This shows that the exponentially growing modes do not contradict the conservation of magnetic moment.

Our conservation property of total magnetic moment $\vec{\mu}$ has few in common with the Bondi-Gold theorem. This theorem (Bondi & Gold, 1950; cf. also Moffatt, 1979, Rädler, 1982, for a discussion of its impact on dynamo theory, and Hollerbach *et al*, 1995, for recent results) guarantees the conservation of the *internal* magnetic moment $\vec{\mu}_{\mathrm{int}} := \dfrac{3}{2\mu_0}\displaystyle\int_{\mathcal{K}_R}\mathbf{B}\,dx'^3$ for a spherical dynamo of radius R surrounded by vacuum in the limit of infinite conductivity within the sphere $\mathcal{K}_R$. Our statement neither involves ideal magnetohydrodynamics anywhere, nor does it prohibit the internal magnetic moment to change, provided that this is compensated by changes in the *external* magnetic moment.

3 A localised dynamo surrounded by vacuum

Sokoloff (1997) suggested that the weak low-frequency electromagnetic wave connected with the time-dependent magnetic field generated by a dynamo surrounded by vacuum compensates for the change in magnetic moment within the dynamo region. To illustrate this idea, we consider a magnetic dipole localised at the origin, which is switched on at time $t{=}0$ in formerly field-free vacuum — say on account of dynamo action. Higher multipole components are of no interest, because their integrals outside a sphere $\mathcal{K}_R$ around the origin exist in the strict sense and vanish altogether.

The general picture of the field distribution is quite obvious: The information about the dipole having appeared at the origin propagates spherically at the speed of light c. In the domain $r{>}ct$ (r being the distance from the dipole), there is still no field, $\mathbf{B}{=}\mathbf{0}$; in the domain $r{<}ct$, there is a stationary dipole field $\mathbf{B}{=}\mathbf{B}_{\mathrm{dip}}$. At the wave front, $r{=}ct$, finally, there is a concentrated, more complex magnetic field, the exact form of which can be obtained by means of retarded potentials, namely

$$\mathbf{B}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\,\mathrm{curl}\int\frac{\mathbf{j}(\mathbf{x}',t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})}{|\mathbf{x}-\mathbf{x}'|}\,dx'^3\ . \tag{15}$$

Note that it is not sufficient to apply the magnetic-dipole approximation of electromagnetic radiation theory, because it is based on the assumption that the wavelength λ is much smaller than the distance r from the emitting source, while in our case the whole continuum of $\{\lambda = 2\pi/k\}$ is important according to the well-known formula for the Fourier transform of the Heaviside function,

$$\theta(r-ct) = \frac{i}{2\pi}\ \mathrm{v.\,p.}\int\limits_{-\infty}^{\infty}\frac{e^{-ik(ct-r)}}{k}dk + \frac{1}{2}\ . \tag{16}$$

Quantitatively, we have a current density

$$\mathbf{j}(\mathbf{x},t) = -\mathbf{m}\times \operatorname{grad}\delta(\mathbf{x})\cdot\theta(t)\ , \tag{17}$$

where $\delta(\mathbf{x})$ denotes the three-dimensional Dirac delta function. Applying Equation (15) to this current field, we obtain (with $r{=}|\mathbf{x}|$)

$$\int \frac{\mathbf{j}(\mathbf{x}',t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})}{|\mathbf{x}-\mathbf{x}'|}\,dx'^3 = -\mathbf{m}\times \operatorname{grad}\frac{\theta(ct-r)}{r} = -\mathbf{m}\times\left(\theta(ct-r)\operatorname{grad}\frac{1}{r} - \frac{\delta(ct-r)}{r^2}\mathbf{x}\right), \tag{18}$$

and finally, after some algebra[2],

$$\begin{aligned}\frac{4\pi}{\mu_0}\mathbf{B}(\mathbf{x},t) \;=\; & \underbrace{\frac{8\pi}{3}\mathbf{m}\,\delta(\mathbf{x})\cdot\theta(t)}_{\mathrm{I}} + \underbrace{\frac{3\mathbf{x}(\mathbf{m}\cdot\mathbf{x})-\mathbf{m}\,r^2}{r^5}[\theta(ct-r)+r\delta(ct-r)]}_{\mathrm{II}} \\ & + \underbrace{\frac{\mathbf{x}(\mathbf{m}\cdot\mathbf{x})-\mathbf{m}\,r^2}{r^3}\delta'(ct-r)}_{\mathrm{III}}\ .\end{aligned} \tag{20}$$

Integrating this over a spherical shell makes term II vanish and we get

$$\frac{3}{2\mu_0}\int_{\mathcal{K}_R}\mathbf{B}(\mathbf{x})\,dx^3 = \underbrace{\mathbf{m}\,\theta(t)}_{\mathrm{I}'} + \underbrace{\mathbf{m}\Big(R\cdot\delta(ct-R) - \theta(R-ct)\cdot\theta(t)\Big)}_{\mathrm{III}'}\ . \tag{21}$$

This means that the whole magnetic moment of the external field is concentrated in the wave front. In other words, the electromagnetic wave that carries the information about the change of the interior dipole moment, also carries away the compensating dipole moment $\vec{\mu}_{\mathrm{ext}}$.

Equations (20) and (21) can easily be generalised to the case of a localised magnetic dipole evolving arbitrarily in time, $\mathbf{j}(\mathbf{x},t) = -\mathbf{m}(t)\times\operatorname{grad}\delta(\mathbf{x})$. This is done by convolving (20) and (21) with $\dot{\mathbf{m}}(t)$, according to the formula

$$\mathbf{m}(t) = \int_{-\infty}^{\infty}\dot{\mathbf{m}}(t-t')\,\theta(t')\,dt' \tag{22}$$

that holds for an arbitrary vector function $\mathbf{m}(t)$ with $\lim\limits_{t\to\infty}\mathbf{m}(t) = \mathbf{0}$.

Returning to Equation (2) as an integral over the total current density $\mathbf{j} = \mathbf{j}_{\mathrm{cond}}+\dot{\mathbf{D}}$, we easily see that the conduction current $\mathbf{j}_{\mathrm{cond}}$ is located at the origin and the displacement current $\dot{\mathbf{D}}$ at the wave front. This again leads us to the interpretation given above.

[2]Note that simply inserting (18) into Equation (15) and exploiting the relation $\Delta 1/r = -4\pi\delta(\mathbf{x})$ would yield a term I equal to $4\pi\mathbf{m}\delta(\mathbf{x})\theta(t)$, i.e. $3/2$ of the right result. This is connected with the non-integrability of dipole fields at the origin and can be avoided by regarding the magnetic field of a current on the surface of a small sphere of radius ε,

$$\mathbf{j} = \frac{3}{4\pi}|\mathbf{m}|\frac{\delta(r-\varepsilon)}{\varepsilon^3}\sin\vartheta\,\mathbf{e}_\varphi \tag{19}$$

(in spherical coordinates $(r,\,\vartheta,\,\varphi)$ with axis parallel to $\mathbf{m}$) and letting $\varepsilon\to+0$.

4 Propagation of magnetic moment in a conducting medium

In the case of plasma of constant conductivity $\sigma_{\rm ext}$ surrounding the dynamo, it is not clear *a priori* that the integral (1) exists even as a principal value, because the magnetic field decays only as $\mathcal{O}(1/r^2)$ for $r{\to}\infty$ (Roberts, 1994; Dobler & Rädler, 1997). We will show that the principal value (1) still exists in this case and obtain a generalisation of the integral (2) for a stationary dynamo surrounded by a conducting medium.

In this section, we will use equations of magnetohydrodynamics, i. e. neglect any displacement currents. This is well justified in the given case of a dynamo acting in a conducting medium that fills the whole space, since it is a very good approximation in all cosmic plasmas (like the medium filling the dynamo discussed in Section 3), cf. for example Moffatt (1978). But now, in contrast to the case of insulating surroundings, the weak "electromagnetic" wave outside $\mathcal{D}$ is determined by the induction equation (4), too. This is because its displacement current is by a factor $\sim U^2/c^2$ weaker than the conduction current (U being a characteristic material velocity) and can therefore be neglected for non-relativistic objects.

A dynamo in a finite region $\mathcal{D}$ of conductivity $\sigma(\mathbf{x})$ (including turbulent diffusion), surrounded by an infinite medium of constant diffusivity $\eta{=}1/(\mu_0\sigma_{\rm ext})$ can be described by the following equations

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} - \eta\Delta\mathbf{B} &= \operatorname{curl} \boldsymbol{\mathcal{F}} && (23)\\ \operatorname{div} \mathbf{B} &= 0\,, && (24)\end{aligned}$$

where we suppose $\boldsymbol{\mathcal{F}}$ to be different from zero only inside $\mathcal{D}$.

In the case of mean-field dynamos, the induced electromotive force $\boldsymbol{\mathcal{F}}$ can be expressed by

$$\boldsymbol{\mathcal{F}} \;=\; \mathbf{u}{\times}\mathbf{B} + \underline{\underline{\alpha}}\mathbf{B} - \tilde{\beta}\operatorname{curl}\mathbf{B}\,, \tag{25}$$

with the "turbulent" diffusivity

$$\tilde{\beta}(\mathbf{x}) = \frac{1}{\mu_0}\left(\frac{1}{\sigma(\mathbf{x})} - \frac{1}{\sigma_{\rm ext}}\right)\;. \tag{26}$$

This *formal* transformation is similar to the standard transformation yielding $\sigma_{\rm turb} = \sigma/(1{+}\mu_0\sigma\beta)$ that can be found in text books on mean-field MHD (e. g. Krause & Rädler, 1980). It allows us to work with the constant diffusivity η in all space, since the deviations of the actual diffusivity from this value that occur within the dynamo region are shifted into the right-hand side of Equation (23).

Beyond the framework of mean-field theory, Equation (25) can not be used. We stress however that *any* induction process in $\mathcal{D}$ can be expressed by Equations (23) and (24), only $\boldsymbol{\mathcal{F}}$ will in general be linked to $\mathbf{B}$ in a more complicated way, e. g. involving the Navier-Stokes equations. A procedure analogous to the above will still allow us to work with the homogeneous diffusivity η in all space.

As was shown by Dobler & Rädler (1997), in the steady case $\partial\mathbf{B}/\partial t{\equiv}0$ Equations (23), (24) are equivalent to the integral equation

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0\sigma_{\rm ext}}{4\pi}\int\limits_{\mathcal{D}} \operatorname{grad}\frac{1}{|\mathbf{x}{-}\mathbf{x}'|}\times\boldsymbol{\mathcal{F}}(\mathbf{x}')\,dx'^3\;. \tag{27}$$

Integrating this over a sphere $\mathcal{K}_R \supset \mathcal{D}$ and making use of the formula

$$\int\limits_{\mathcal{K}_R} \operatorname{grad} \frac{1}{|\mathbf{x}-\mathbf{x}'|}\, dx^3 = \frac{4\pi}{3}\mathbf{x}' \qquad \text{for } R > |\mathbf{x}'| \,, \tag{28}$$

we find that

$$\frac{3}{2\mu_0} \int\limits_{\mathcal{K}_R} \mathbf{B}(\mathbf{x})\, dx^3 = \frac{1}{2}\sigma_{\text{ext}} \int\limits_{\mathcal{D}} \mathbf{x}' \times \boldsymbol{\mathcal{F}}(\mathbf{x}')\, dx'^3 \,. \tag{29}$$

As the integral on the right is independent from R, we find that it is equal to $\vec{\mu}$ as defined by (1). The magnetic moment thus exists and moreover can be calculated by an integral over the bounded region $\mathcal{D}$.

It is not difficult to see that (29) contains Equation (2) as the limit for $\sigma_{\text{ext}} \to \infty$, keeping $\sigma(\mathbf{x})$ fixed. Therefore, Equation (29) is really a generalisation of the classical definition (2).

Let us now return to the time-dependent case. In the case of a dynamo embedded into a conducting medium, the magnetic moment $\vec{\mu}_{\text{ext}}$ propagates towards infinity in the form of neither electromagnetic, nor dynamo wave. Rather, it is transported diffusively, i. e. magnetic field that is subject to diffusion with transport coefficient $\eta = 1/(\mu_0 \sigma_{\text{ext}})$ carries it away.

This statement can be substantiated with the following formulas. With the help of Green's function of the left-hand side of Equation (23)

$$G(\mathbf{x}, t) = \frac{1}{(4\pi\eta\, t)^{\frac{3}{2}}} e^{-\frac{\mathbf{x}^2}{4\eta\, t}} \tag{30}$$

we can write the solution $\mathbf{B}(\mathbf{x}, t)$ of Equations (23), (24) in the form of an integral equation

$$\begin{aligned} \mathbf{B}(\mathbf{x}, t) \;\; = & \int\limits_{\mathbb{R}^3} dx'^3 G(\mathbf{x}-\mathbf{x}', t) \mathbf{B}(\mathbf{x}', 0) \\ & - \int\limits_0^t dt' \int\limits_{\mathcal{D}} dx'^3 \;\; \operatorname{grad}' G(\mathbf{x}-\mathbf{x}', t-t') \times \boldsymbol{\mathcal{F}}(\mathbf{x}', t') \,, \end{aligned} \tag{31}$$

cf. Dobler & Rädler (1997).

Integrating over a sphere $\mathcal{K}_R$ of radius R centred at the origin, we get

$$\begin{aligned} \int\limits_{\mathcal{K}_R} \mathbf{B}(\mathbf{x}, t)\, dx^3 \;\; = & \int\limits_{\mathbb{R}^3} dx'^3 \underbrace{\int\limits_{\mathcal{K}_R} dx^3\, G(\mathbf{x}-\mathbf{x}', t)}_{1 - \mathcal{O}\left(e^{-\frac{(R-r')^2}{4\eta\, t}}\right)} \; \mathbf{B}(\mathbf{x}', 0) \; + \\ & + \int\limits_0^t dt' \int\limits_{\mathcal{D}} dx'^3 \underbrace{\int\limits_{\partial\mathcal{K}_R} G(\mathbf{x}-\mathbf{x}', t-t')\, d\mathbf{f}}_{\substack{\sim \frac{1}{\sqrt{4\pi\eta\, t}} \frac{R}{r'} \frac{\mathbf{x}'}{r'} e^{-\frac{(R-r')^2}{4\eta\, t}} \\ \text{for } R \gg (r', \sqrt{4\eta\, t})}} \times \boldsymbol{\mathcal{F}}(\mathbf{x}', t') \,. \end{aligned} \tag{32}$$

Letting $R\to\infty$, it becomes evident that generation ($\mathcal{F}$) produces magnetic field with net magnetic moment $\equiv\mathbf{0}$,[3] and the magnetic moment $\vec{\mu}$ of the whole field is due to the initial field $\mathbf{B}(\mathbf{x},0)$, which propagates diffusively.

5 Conclusion

We have shown that, under certain assumptions, the total magnetic moment $\vec{\mu}$ defined by (1) is a conserved quantity for dynamo systems and generally evolving current systems. We have demonstrated that this conservation law is nevertheless compatible with dynamo action in a bounded region surrounded by an either insulating or homogeneously conducting medium. We stress once more that the considerations of the rate of change of $\vec{\mu}$ given by Moffatt (1978) and related to the Bondi-Gold theorem (Bondi & Gold, 1950) concern the *internal* magnetic moment $\vec{\mu}_{\rm int}$ only, since they left the displacement currents due to electromagnetic wave propagation out of consideration. $\vec{\mu}_{\rm int}$ *can* change with time and this is why dynamo processes are not prohibited by our conservation law.

In the case of stationary currents, $\partial\mathbf{j}/\partial t\equiv\partial\mathbf{B}/\partial t\equiv\mathbf{0}$, the magnetic moment is in general not zero but rather given by the integral (1) or its variants (2) (with $\mathbf{j}\equiv\mathbf{j}_{\rm cond}$) or (29). This indicates that such a saturated field can not be produced by dynamos (or other physical processes) from a very weak seed field with negligible magnetic moment $\vec{\mu}$. But this is not paradoxical as it might seem at a first glance. The difference between a really stationary field and an almost stationary one produced by long-lasting dynamo action is substantial only far away from the dynamo region $\mathcal{D}$, outside some "sphere of influence" of the dynamo history, at $r\gtrsim R_{\rm infl}$. Suppose that the dynamo reached a steady regime within $\mathcal{D}$ at time t_0 and denote the magnetic flux density generated by this dynamo system by $\mathbf{B}(\mathbf{x},t;t_0)$. For surrounding vacuum, we have $R_{\rm infl}=c(t-t_0)$, since within this radius the fields are strictly steady. In a homogeneously conducting medium, Equation (32) gives $R_{\rm infl}\approx\sqrt{\eta(t-t_0)}$. The apparent paradox just mentioned is due to the illegal interchange of two limits: the limit $R\to\infty$ in the definition (3) of the principal value, and the limit $t_0\to-\infty$ to obtain a strictly stationary field. The two quantities

$$\vec{\mu}_1 \;:=\; \lim_{t_0\to-\infty}\lim_{R\to\infty}\int_{\mathcal{K}_R}\mathbf{B}(\mathbf{x},t;t_0)\,dx^3 \;=\; \lim_{t_0\to-\infty}\vec{\mu}(t;t_0) \tag{33}$$

$$\vec{\mu}_2 \;:=\; \lim_{R\to\infty}\int_{\mathcal{K}_R}\lim_{t_0\to-\infty}\mathbf{B}(\mathbf{x},t;t_0)\,dx^3 \tag{34}$$

take different values in our scenario where a "stationary" magnetic field is produced from an initial field with zero magnetic moment: While $\vec{\mu}_1=\mathbf{0}$ since $\vec{\mu}$ is conserved and $\vec{\mu}(t_0;t_0)=\mathbf{0}$, the second quantity $\vec{\mu}_2$ corresponds to the value given by Equation (1) for the infinitely extended stationary field. It depends on the given situation (and the questions to be investigated), which of the quantities $\vec{\mu}_1$ and $\vec{\mu}_2$ is the "better" one. According to our conservation law from Section 2, both of them are conserved.

[3] This is quite reasonable, because in a conducting space the newly generated field lines are essentially closed within a finite region and $\int_{\mathcal{T}}\mathbf{B}\,dx^3$ over a closed magnetic flux tube $\mathcal{T}$ is equal to zero.

Acknowledgements

I am grateful to D. Sokoloff for stimulating discussions and to K.-H. Rädler for helpful comments.

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