INTEGRAL EQUATIONS FOR KINEMATIC DYNAMO MODELS

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Summary: *A new technique for the treatment of the kinematic dynamo problem is presented. The method is applicable when the dynamo is surrounded by a medium of finite conductivity and is based on a reformulation of the induction equation and boundary conditions at infinity into an integral equation.*

We show that the integral operator $\hat{I}$ involved here is compact in the case of homogeneous conductivity, which is important for both mathematical and numerical treatment. A lower bound for the norm of $\hat{I}$ then yields a necessary condition for the generation of magnetic fields by kinematic dynamos.

Numerical results are presented for some simple $\alpha^2\omega$-dynamo models.

The far-field asymptotics for stationary and time-dependent field modes is discussed.

Key words: $\alpha^2\omega$-dynamos, kinematic dynamos, Biot-Savart law, far field

1. BIOT-SAVART LAW AND DYNAMO THEORY

Consider a bounded dynamo region surrounded by a medium of non-vanishing electrical conductivity. In the case of steady magnetic fields and constant electrical conductivity in the whole space, $\sigma \equiv \mathrm{const}$, one usually inserts Ohm's law

$$\mathbf{j} = \sigma(\mathbf{E} + \boldsymbol{\mathcal{F}}) \ , \tag{1}$$

into Maxwell's equations, which leads to the well-known induction equation

$$\frac{\partial \mathbf{B}}{\partial t} - \Delta \mathbf{B} = C \,\mathrm{curl}(\mathbf{u}\times\mathbf{B} + \alpha\mathbf{B} - \beta \,\mathrm{curl}\,\mathbf{B}) \ , \qquad \mathrm{div}\,\mathbf{B} = 0 \ , \tag{2}$$

given here in standard non-dimensional form. Alternatively, we can we apply the Biot-Savart law

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_0}{4\pi} \int \frac{(\mathbf{x}-\mathbf{x}')\times\mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \, dx'^3 \tag{3}$$

$$= \frac{\mu_0}{4\pi} \int \frac{\mathrm{curl}'\,\mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \, dx'^3 \ . \tag{4}$$

to Ohm's law (1). The second form (4) is obtained by means of integration by parts and shows that an arbitrary gradient can be added to or subtracted from $\mathbf{j}(\mathbf{x})$

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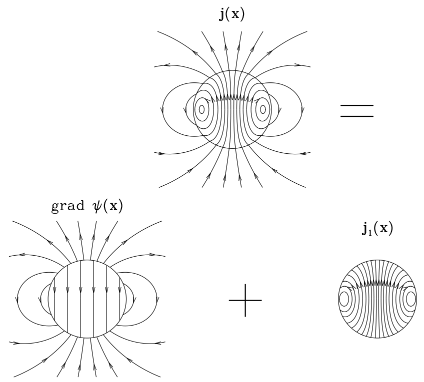

Fig. 1. Illustration of the reduction that leads to Formula (5). The total current field $\mathbf{j}$ is split up into a gradient, grad ψ, and a vector field $\mathbf{j}_1$ with finite support. Since the gradient does not contribute to the integral (3), the latter is reduced to an integral over the dynamo region $\mathcal{D}$ (the unit sphere) only.

without changing any of the two integrals (3) or (4). This is illustrated in Figure 1 for the case of a simple spherical dynamo model.

Applying (3) to the current field (1) we find that, for $\sigma \equiv \text{const}$, the term $\sigma\mathbf{E}$ does not contribute to the Biot-Savart integral (3) since it can be written as a gradient, grad$(\sigma\Phi)$. In non-dimensional variables we thus get the integral equation

$$\left(\hat{I}\mathbf{B}\right)(\mathbf{x}) := \int_{\mathcal{D}} \frac{(\mathbf{x}-\mathbf{x}')\times\boldsymbol{\mathcal{F}}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\, dx'^3 = -\frac{4\pi}{C}\mathbf{B}(\mathbf{x}) \ . \tag{5}$$

The induced electromotive force $\boldsymbol{\mathcal{F}}$ is usually given by mean-field electrodynamics (cf. *Krause and Rädler, 1980*) and we will discuss the linear form

$$\boldsymbol{\mathcal{F}} = \mathbf{u}\times\mathbf{B} + \alpha\mathbf{B} - \beta\,\mathrm{curl}\,\mathbf{B} \ , \tag{6}$$

with coefficients $\mathbf{u}$, α (which may be a tensor), β that are given functions of the coordinate vector $\mathbf{x}$ alone. However, (5) holds (as a nonlinear integral equation) also in the nonlinear case.

Together with (6), Equation (5) represents a linear integral-eigenvalue problem. The stationary magnetic field modes $\mathbf{B}(\mathbf{x})$ are given by the eigenfunctions, while the corresponding critical dynamo numbers $C = \mu_0\sigma UL$ are given by the eigenvalues of the operator $\hat{I}$. Here U is a velocity characteristic for velocity field $\mathbf{u}(\mathbf{x})$ and α-effect $\alpha(\mathbf{x})$, and L is a characteristic length of the dynamo region. The integral equation (5) was first given by *Roberts (1967, 1994)* for laminar dynamos with $\alpha\equiv 0$, $\beta\equiv 0$.

Note that the term containing β allows us to apply (5) to cases where physical conductivity is constant only outside the dynamo region $\mathcal{D}$; cf. *Dobler and Rädler (1998a)* for details.

We will speak of *homogeneous conductivity* if $\beta(\mathbf{x}){\equiv}0$. In that case our integral equation takes the form

$$\int\limits_{\mathcal{D}} \frac{(\mathbf{x}-\mathbf{x}')\times[\mathbf{u}(\mathbf{x}')\times\mathbf{B}(\mathbf{x}') + \alpha(\mathbf{x}')\mathbf{B}(\mathbf{x}')]}{|\mathbf{x}-\mathbf{x}'|^3}\, dx'^3 = -\frac{4\pi}{C}\mathbf{B}(\mathbf{x}) \tag{7}$$

and the operator $\hat{I}$ from (5) is an integral operator with *weak singularity* at $\mathbf{x}'{=}\mathbf{x}$. Therefore, and because the domain of integration is bounded, $\hat{I}$ is bounded and, moreover, *compact* (cf. *Kress, 1989*, theorem 2.21). Hence, Riesz' first theorem (theorem 3.1 or particularly 3.11 in the book of Kress) tells us immediately that the spectrum of $\hat{I}$ is countable (i. e. discrete) and has no other point of accumulation than 0 (corresponding to $C{=}\infty$).

2. NUMERICAL RESULTS

In this section we present some results obtained by discretising Equation (7), which holds for *steady, axisymmetric modes* of $\alpha^2\omega$-dynamos in the case of *homogeneous conductivity.* In cylindrical coordinates (ϱ, φ, z) and for axisymmetric fields (and axisymmetric induction effects), Equation (5) can be written in the form

$$-\frac{4\pi}{C}B_\varrho = -\hat{A}B_\varphi \tag{8a}$$

$$-\frac{4\pi}{C}B_\varphi = (\hat{A}{+}\hat{F})B_\varrho + (\hat{D}{+}\hat{G})B_z \tag{8b}$$

$$-\frac{4\pi}{C}B_z = \hat{E}B_\varphi\ , \tag{8c}$$

with certain integral operators $\hat{A}$, $\hat{D}$, $\hat{E}$ (representing the α-effect) and $\hat{F}$, $\hat{G}$ (for differential rotation) that are explicitely given by *Dobler & Rädler (1998a).*

From the system (8) we can eliminate B_ϱ and B_z and get an integral equation in B_φ alone,

$$\Big[(\hat{D}{+}\hat{G})\hat{E} - (\hat{A}{+}\hat{F})\hat{A}\Big]B_\varphi = \left(\frac{4\pi}{C}\right)^2 B_\varphi\ . \tag{9}$$

The results given below are obtained by discretising (9) and solving the resulting matrix eigenvalue problem by numerical standard techniques.

2.1 SPHERICAL MODEL

First we considered the spherical α^2-dynamo model

$$\alpha(\mathbf{x}) = \begin{cases} \alpha_0\cos\vartheta\ , & r<R \\ 0\ , & r>R \end{cases} \quad ; \qquad \omega\equiv 0\ , \tag{10}$$

where $r{=}|\mathbf{x}|$, $\cos\vartheta = z/\sqrt{\varrho^2{+}z^2}$.

For surrounding vacuum, *Roberts (1972)* found the first two modes to be a dipole ($C_{\text{crit}} = 7.641$) and a quadrupole ($C_{\text{crit}} = 7.808$). Our results for homogeneous conductivity are given in Table 1 and show that all eigenmodes appear in dipole-quadrupole pairs of equal critical dynamo number, a phenomenon we will refer to as *dipole-quadrupole degeneration.*

Table 1. Critical dynamo numbers for the dynamo (10).

N	C					
394	6.73345	6.73345	10.5839	10.5839	11.3624	11.3624

This degeneration is related to *Roberts' (1960)* adjointness theorem and has been proven by *Proctor (1977a, 1977b).* It appears in a rather broad class of kinematic dynamo systems. In a subsequent paper (*Dobler and Rädler, 1998a*) we use our integral-equation formalism to show, inspired by Proctor's proof, that for homogeneous conductivity
a) if α is antisymmetric with respect to the equatorial plane, any α^2-dynamo shows dipole-quadrupole degeneration and
b) if, in addition, a velocity-field exists that is mirror-symmetric with respect to the equatorial plane, then with every dipole mode to the velocity field $\mathbf{u}$ there exists a quadrupole mode to the reversed velocity field, $-\mathbf{u}$, and vice versa.

2.2 ELLIPTIC MODELS

Next, we examined an $\alpha^2\omega$-dynamo in an oblate spheroid $\varrho^2/a^2 + z^2/b^2 < 1$ for different aspect ratios $a/b \in \{1, 3\}$. For $\alpha(\mathbf{x})$ and $\omega(\mathbf{x})$ we chose

$$\alpha(\mathbf{x}) = \begin{cases} \alpha_0 \dfrac{z}{b} \ , & \varrho^2/a^2 + z^2/b^2 < 1 \\ 0 \ , & \text{otherwise} \end{cases} \tag{11}$$

$$\omega(\mathbf{x}) = \frac{C_\omega}{C_\alpha} \frac{\alpha_0}{a^2} \cdot \begin{cases} (\varrho - a) \ , & \varrho^2/a^2 + z^2/b^2 < 1 \\ \text{linear (in } r \text{) to zero} \ , & 1 < \varrho^2/a^2 + z^2/b^2 < 4 \\ 0 \ , & \text{otherwise} \end{cases} \tag{12}$$

Table 2. Critical dynamo numbers for the α^2-dynamo (11), (12), C_ω=0 with different aspect ratios a/b.

a/b	N	C_α					
1	366	9.54674	9.54674	14.3614	14.3614	18.2074	18.2074
3	344	18.2931	18.2931	25.3462	25.3462	32.3976 $\pm 0.34277i$	32.3976 $\pm 0.34277i$

For both ellipsoids, two different values of $|C_\omega/C_\alpha| \in \{0,1\}$ were examined. For $C_\omega = 0$, the α^2-dynamo, we again have dipole-quadrupole degeneration, i. e. dipole and quadrupole modes have equal conditions of excitation, as can be seen in Table 2. Table 3 shows the critical dynamo numbers for $C_\omega/C_\alpha = \pm 1$. Now, differential rotation breaks dipole-quadrupole degeneration. The generalised degeneration [item b) above], however, makes Table 3 valid for positive as well as negative C_ω/C_α.

Table 3. Critical dynamo numbers for the $\alpha^2\omega$-dynamo (11), (12), $C_\omega/C_\alpha = \pm 1$. The first mode is a dipole mode (shown in the left half of Figures 2, 3) for sign "+" and a quadrupole mode (right half of Figures 2, 3) for sign "−".

a/b	N	C_α					
1	366	9.38634	9.73792	14.0924	14.6648	18.1471	18.3397
3	344	17.4321	19.1533	24.3266	26.7715	31.5100	32.6088

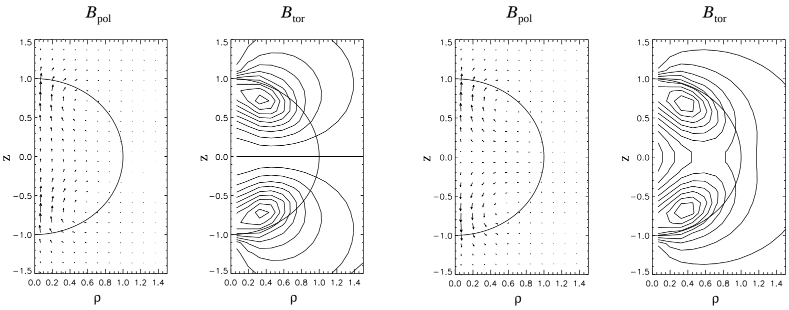

Fig. 2. The first mode for the "ellipsoid" dynamo (11), (12) for a=b=1. Left half: $C_\omega/C_\alpha = +1$; right half $C_\omega/C_\alpha = -1$. Both modes correspond to $C_\alpha = 9.386$.

The complex dynamo numbers in Tables 2, 3 probably indicate oscillating modes, whereby a conjugate complex pair of eigenvalues represents only *one* field mode. See Section 3 for a discussion of time-dependent modes.

3. TIME DEPENDENT MAGNETIC FIELD

For fields with time-depence of the form $\mathbf{B}(\mathbf{x},t) = \tilde{\mathbf{B}}(\mathbf{x})e^{\gamma t}$ with complex $\gamma \notin \mathbb{R}^-$, one can derive the integral equation (cf. *Dobler and Rädler, 1998a*)

$$\left(\hat{I}^{(\gamma)}\mathbf{B}\right)(\mathbf{x}) = \int \frac{(\mathbf{x}-\mathbf{x}')\times\boldsymbol{\mathcal{F}}}{|\mathbf{x}-\mathbf{x}'|^3}\, e^{-\sqrt{\gamma}|\mathbf{x}-\mathbf{x}'|}\left(1+\sqrt{\gamma}|\mathbf{x}-\mathbf{x}'|\right) dx'^3 = -\frac{4\pi}{C}\mathbf{B}(\mathbf{x})\ , \quad (13)$$

where the complex square root is chosen such that Re $\sqrt{z} \geq 0\ \forall z$. For homogeneous conductivity the operator $\hat{I}^{(\gamma)}$ is again compact.

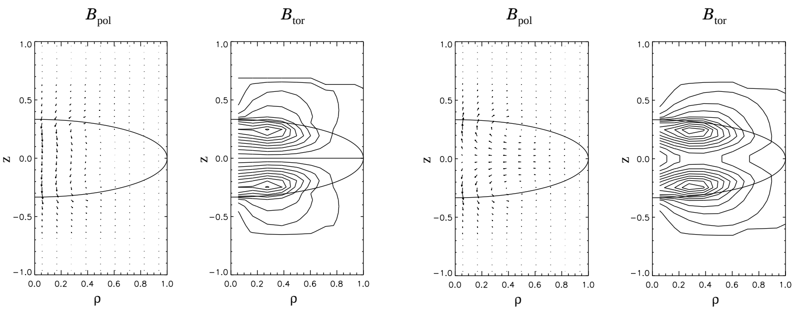

Fig. 3. Same as Figure 2, but for $a=1$, $b=1/3$. Both modes correspond to $C_\alpha = 9.008$.

The eigenvalues $1/C$ of (13) are in general complex quantities. If we fix $\operatorname{Re}\gamma$ (say, $\operatorname{Re}\gamma = 0$ to investigate the marginal case), C will become real — and thus physically meaningful — only for certain values of $\Omega := \operatorname{Im}\gamma$. In a numerical application of (13) one would thus have to scan for the roots Ω_* of the functions $\operatorname{Im} C(\Omega)$ in order to get the oscillation frequencies Ω_* and dynamo numbers $C(\Omega_*)$ of the corresponding magnetic field modes $\mathbf{B}(\mathbf{x})$.

4. A NECESSARY CONDITION FOR MAGNETIC FIELD GENERATION

We already mentioned that the integral operators in Equations (5) and (13) are compact, and thus bounded, for homogeneous conductivity. A concrete upper bound for the norm $\|\hat{I}^{(\gamma)}\|$ is (*Dobler and Rädler 1998a*)

$$\|\hat{I}^{(\gamma)}\| \leq 2\pi L_{\mathcal{D}} \left(\|\mathbf{u}\|_\infty + \|\alpha\|_\infty\right) \ . \tag{14}$$

As a corollary we obtain an estimate for the dynamo numbers C of steady (linear or nonlinear) and time dependent (with $\gamma \notin \mathbb{R}^-$) dynamos with homogeneous conductivity:

$$|C| = \mu_0 \sigma L |U| \geq \frac{2L}{L_{\mathcal{D}}} \frac{U}{\|\mathbf{u}\|_\infty + \|\alpha\|_\infty} \ . \tag{15}$$

This is a *necessary condition* for the excitation of magnetic field by mean-field dynamos with homogeneous conductivity. It has first been derived by *Roberts (1967, 1994)* for the steady case.

In the case of constant electrical conductivity and solenoidal motions, the necessary condition of *Childress (1969)* reads

$$\mu_0 \sigma R \|\mathbf{u}\|_\infty \geq \mu_0 \sigma R \frac{|\Delta_{\max}\mathbf{u}|}{2} \geq \frac{\pi}{2} \ ,$$

where $\Delta_{\max}\mathbf{u}$ is the maximum relative velocity inside the sphere. Our estimate (15) yields the weaker result $|C_{\text{crit}}| = \mu_0 \sigma R \|\mathbf{u}\|_\infty \geq 1$.

5. THE FAR FIELD

One result that can be easily derived from the integral equation (5) is the asymptotic behaviour of the steady field **B** far from the dynamo region:

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_0 \sigma}{4\pi} \frac{\mathbf{x}}{|\mathbf{x}|^3} \times \int \boldsymbol{\mathcal{F}}(\mathbf{x}')\, dx'^3 + \mathcal{O}(1/r^3) \tag{16}$$

$$= \frac{\mu_0}{4\pi} L |\mathbf{I}_1| \frac{\sin\vartheta}{r^2} \mathbf{e}_\varphi + \mathcal{O}(1/r^3) \qquad \text{for } r = |\mathbf{x}| \to \infty \tag{17}$$

(in dimensional form), where (r, ϑ, φ) represent polar coordinates whose z-axis is chosen parallel to the vector $L\mathbf{I}_1 := \sigma \int \boldsymbol{\mathcal{F}}(\mathbf{x}')\, dx'^3$.

For most cosmic dynamos, $\alpha(\mathbf{x})$ is antisymmetric and $\mathbf{u}(\mathbf{x})$ and $\beta(\mathbf{x})$ are symmetric with respect to the equatorial plane. In that case, we find that the first mode with quadrupole symmetry generally decays slower ($|\mathbf{B}^{\mathcal{S}}(\mathbf{x})| \sim 1/r^2$) than the first dipole-like mode ($|\mathbf{B}^{\mathcal{A}}(\mathbf{x})| \sim 1/r^3$) for $r \to \infty$. In this light, the use of the terms "quadrupole-like / dipole-like" for antisymmetric and symmetric fields seems quite questionable when the fields do not allow for multipole expansion.

For time-dependent fields with $\gamma \notin \mathbb{R}^-$, Equation (13) shows that the far field decays exponentially in r.

Manuscript received: 31st January, 1998; *Revisions accepted: 15th May 1998*

References

Childress, S., 1969: Théorie magnétohydrodynamique de l'effet dynamo. *Report from Département Méchanique de la Faculté des Sciences*, Paris

Dobler, W. and Rädler, K.-H., 1998a: An integral equation approach to kinematic dynamo models. *Geoph. Astrop. Fluid Dyn.* (submitted)

Dobler, W. and Rädler, K.-H., 1998b: Dipole-quadrupole degeneration in kinematic dynamo models with homogeneous conductivity. *(in preparation)*

Krause, F. and Rädler, K.-H., 1980: *Mean-Field Magnetohydrodynamics and Dynamo Theory.* Akademie-Verlag/Pergamon Press, Berlin-Oxford

Kress, R., 1989: *Linear Integral Equations.* Springer, Heidelberg

Proctor, M.R.E., 1977a: On the eigenvalues of kinematic α-effect dynamos. *Astron. Nachr.*, **298**, pp. 19-25

Proctor, M.R.E., 1977b: The role of mean circulation in parity selection by planetary magnetic fields. *Geoph. Astrop. Fluid Dyn.*, **8**, pp. 311-324

Roberts, P.H., 1960: Characteristic value problems posed by differential equations arising in hydrodynamics and hydromagnetics. *J. Math. Anal. & Applic.*, **1**, pp. 195-214

Roberts, P.H., 1967: *An Introduction to Magnetohydrodynamics.* Longmans, London

Roberts, P.H., 1972: Kinematic Dynamo Models. *Philos. Trans. R. Soc. Lond.*, **A272**, pp. 663-703

Roberts, P.H., 1994: Fundamentals of dynamo theory, In: *Lectures on Solar and Planetary Dynamos*, Eds. M.R.E. Proctor and A.D. Gilbert, Cambridge University Press, pp. 1-58