# An Integral Equation Approach to Kinematic Dynamo Models


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**Abstract**

The paper deals with dynamo models in which the induction effects act within a bounded region surrounded by an electrically conducting medium at rest. Instead of the induction equation, an equivalent integral equation is considered, which again poses an eigenvalue problem. The eigenfunctions and eigenvalues represent the magnetic field modes and corresponding dynamo numbers.

In the simplest case, that is for homogeneous conductivity and steady fields, this integral equation follows immediately from the Biot-Savart law. For this case, numerical results are presented for some spherical and elliptical axisymmetric $\alpha^2\omega$-dynamo models. For a large class of models the interesting feature of dipole-quadrupole degeneration is found.

Using Green's function of a Helmholtz-type equation, we derive a more general integral equation, which applies to time-dependent magnetic field modes, too, and gives us some insight into the spectral properties of the integral operators involved. In particular, for homogeneous conductivity the operator is compact and thus bounded, which leads to a necessary condition for dynamo action.

KEY WORDS: *$\alpha^2\omega$-dynamos, kinematic dynamos, mean-field electrodynamics*

# 1 Introduction

Most cosmic magnetic fields (like those of planets, the Sun and galaxies) are believed to be generated and maintained against dissipation by dynamo mechanisms. In many astrophysical objects, turbulent motions play a crucial rôle and consequently the electromagnetic fields show turbulent features, too. We will therefore adopt the mean-field concept (see e. g. Krause and Rädler, 1980) that proved to be useful in this context. The original equations of magnetohydrodynamics can be easily recovered as a special case, putting to zero the coefficients describing turbulence effects.

The mean-field dynamo problem as we will discuss it here is defined by Maxwell's equations (in MHD approximation)

$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1. a}$$

$$\operatorname{div}\mathbf{B} = 0 \tag{1. b}$$

$$\operatorname{curl}\mathbf{B} = \mu_0 \mathbf{j} \tag{1. c}$$

and Ohm's law

$$\mathbf{j} = \sigma(\mathbf{E} + \boldsymbol{\mathcal{F}}) \; . \tag{2}$$

The electromotive force $\boldsymbol{\mathcal{F}}$ is due to the mean and turbulent motions. We will use the mean-field relation

$$\boldsymbol{\mathcal{F}} = \mathbf{u}\times\mathbf{B} + \underline{\underline{\alpha}}\mathbf{B} - \beta \operatorname{curl}\mathbf{B} \; , \tag{3}$$

but in principle all the equations given below that involve $\boldsymbol{\mathcal{F}}$ hold independent of this particular form. $\mathbf{B}$, $\mathbf{E}$, $\mathbf{j}$ and $\mathbf{u}$ are magnetic flux density, electric field strength, electric current density and velocity field of the fluid, all understood as mean quantities. We assume the fluid to have the magnetic permeability $\mu_0$ of vacuum; $\sigma$ denotes the electrical conductivity; $\underline{\underline{\alpha}}$ the tensor describing the $\alpha$-effect. The parameter $\beta$ characterises turbulent dissipation of the magnetic field and is, for sake of simplicity, supposed to be scalar. We suppose that the induction effects represented by $\mathbf{u}$, $\underline{\underline{\alpha}}$ and $\beta$ act only within a bounded region $\mathcal{D}$, which we will call *dynamo region*. Both $\mathcal{D}$ and all infinite space surrounding it are supposed to be electrically conducting, i. e. $\sigma>0$ everywhere.

The fields $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{j}$ are supposed to be square integrable. Note that the square integrability of $\mathbf{B}$, i. e. $\int \mathbf{B}^2\, dx^3 < \infty$, implies the absence of currents at infinity. In connection with the equation $\Delta\mathbf{B}{=}\mathbf{0}$ holding outside the bounded dynamo region for steady fields and homogeneous conductivity, it also excludes magnetic flux from infinity. This can be seen from the asymptotics (14) in Section 2.2, that has the property $B_r{\cdot}r^2 \to 0$ for $r{=}|\mathbf{x}|{\to}\infty$. These two physical requirements replace the assumption that $\mathbf{B}$ decays like a dipole field, which holds only in the case of a dynamo surrounded by an insulator.

In this paper, we are interested mostly in the *kinematic dynamo problem*, where the coefficients $\mathbf{u}$ and $\underline{\underline{\alpha}}$ in Equation (3) are given functions of position and the resulting problem is linear in $\mathbf{B}$. Actually, most of our discussion remains valid in the more general case $\mathbf{u} = \mathbf{u}(\mathbf{x};\mathbf{B})$, $\underline{\underline{\alpha}} = \underline{\underline{\alpha}}(\mathbf{x};\mathbf{B})$, $\beta = \beta(\mathbf{x};\mathbf{B})$, and we will mention the relation with this case where it seems necessary.

The traditional approach to the dynamo problem is based on the induction equation, which can be easily derived from Equations (1)–(3). For the case of homogeneous conductivity, $\sigma{\equiv}const$ (everywhere), it reads

$$\frac{\partial\mathbf{B}}{\partial t} - \Delta\mathbf{B} = C\,\mathrm{curl}(\mathbf{u}{\times}\mathbf{B} + \underline{\underline{\alpha}}\mathbf{B} - \beta\,\mathrm{curl}\,\mathbf{B})\ , \qquad \mathrm{div}\,\mathbf{B} = 0\ . \tag{4}$$

As usual, we have introduced here dimensionless variables based on a unit length $L$ (typically the extension of the dynamo region $\mathcal{D}$) and a time $\mu_0\sigma L^2$ (the diffusion time). $\mathbf{u}$ and $\alpha$ are measured in a unit velocity $U$ (typically the maximum value of $|\mathbf{u}|$ or $\|\underline{\underline{\alpha}}\|$) and $\beta$ in $U{\cdot}L$. We have not merged the term $\mathrm{curl}\,\beta\,\mathrm{curl}\,\mathbf{B}$ with the Laplacian on the left hand side; this is useful for our later discussion of the case of non-homogeneous conductivity. Finally, we have introduced the *dynamo number*

$$C = \mu_0\sigma UL\ , \tag{5}$$

which is a dimensionless measure of the strength of the induction effects compared to Ohmic dissipation.

On this level, the dynamo problem consists in finding solutions $\mathbf{B}$ of (4) which do not decay as $t \to \infty$ in the sense that $\|\mathbf{B}\|_2{:=}\int\mathbf{B}^2\,dx^3\ >\ K\ \forall\, t$ with a fixed lower bound $K > 0$. In particular, we will look for solutions of time-dependence $\sim\exp(\gamma t)$. Then, the induction equation (4) poses an eigenvalue problem for the complex growth rate $\gamma = \gamma_{\mathrm{r}}{+}i\gamma_{\mathrm{i}}$, or, if we are interested in solutions with given $\gamma_{\mathrm{r}}$, a two-parameter eigenvalue problem for $C$ and $\gamma_{\mathrm{i}}$. We call *critical dynamo numbers* the values of $C$, for which there exist such solutions with $\mathfrak{Re}\,\gamma = 0$.

Several methods have been used to solve the dynamo problem at different levels of idealisation, the most common approaches being spectral methods, finite-difference methods and asymptotic analysis. The majority of the published models supposes vacuum outside the dynamo region, and their results cannot be directly compared with ours.

Finite-difference methods have to cope with non-local boundary conditions at the outer boundary of the finite volume of calculation (cf. Bräuer and Rädler, 1986, for the case of vacuum surrounding the dynamo). In order to overcome this difficulty, one usually inflates the domain of calculation considerably, which increases numerical cost. When a Cauchy problem is solved instead of an eigenvalue problem, only the leading mode (or a few leading ones) can be calculated and no understanding of the overall spectral structure is obtained. There are, however, also finite-difference models that solve for the whole spectrum of the kinematic dynamo problem.

In this paper, we propose another approach to solve the dynamo problem. Instead of reducing Equations (1), (2) to a eigenvalue problem for a differential operator we transform it into an eigenvalue problem for an integral operator, which we then solve numerically. Such an integral-eigenvalue equation is the continuous analog of a matrix eigenvalue problem. The simplest version of this equation for $\mathbf{B}(\mathbf{x})$ is given by Equation (12) below and holds in the case of homogeneous conductivity and steady electromagnetic fields. Solving it on the bounded region $\mathcal{D}$ has several advantages compared to the induction equation (4). First, the boundary conditions are automatically fulfilled, due to the very construction of the equation. Moreover, the action of the whole current system in the conducting medium outside the dynamo region is contained in (12) in a consistent way, but its explicit treatment is unnecessary: we can solve for the field $\mathbf{B}(\mathbf{x})$ in $\mathcal{D}$ alone.

Our assumption that $\sigma$ be constant in the whole space surrounding the dynamo region is somewhat opposite to the (mostly vacuum surrounded) traditional dynamo models. For stellar and galactic dynamos, however, the surrounding space is also a conductor, and our assumption seems not less realistic than assuming vacuum. Moreover, vacuum poses some conceptual problems in magnetohydrodynamics because quasi-stationarity is well fulfilled in all cosmic plasmas, but not in vacuum (cf. Sokoloff, 1997). One charactersation of quasi-stationary systems is that the electromagnetic fields propagate at a time scale much shorter than both, the material advection time $\tau_{\rm ad}{=}L/U$, and the diffusion time $\tau_{\rm diff}{=}\mu_0\sigma L^2$. But for $\sigma{\to}0$, the becomes infinitely small. Thus, dynamo models embedded into conducting space have relevance to both, applications and fundamental aspects of dynamo theory.

In Section 2 we derive the integral equation for the case of steady fields and obtain the far-field asymptotics for steady dynamos surrounded by a conducting medium. In Section 3 we present numerical results for some $\alpha^2\omega$-dynamo models. Section 4 is devoted to the mathematical foundation of the integral equation and its generalisation to the time-dependent case. A necessary condition for self-excited dynamos with homogeneous conductivity is established and a generalised integral equation for varying conductivity is given.

# 2 The Biot-Savart law and the integral eigenvalue equation for B

## 2.1 The integral equation

For a given square integrable current field $\mathbf{j}$, Equations (1.b) and (1.c) determine $\mathbf{B}$ in the following way (cf. for example Jackson, 1980):

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_0}{4\pi}\int \frac{(\mathbf{x}-\mathbf{x}')\times\mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\,dx'^3 \tag{6}$$

$$= \frac{\mu_0}{4\pi}\int \frac{\mathrm{curl}'\,\mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,dx'^3 \;. \tag{7}$$

Equation (6) is the well-known Biot-Savart law; (7) is obtained by integration by parts* and shows that an arbitrary gradient can be added to or subtracted from $\mathbf{j}(\mathbf{x})$ without changing any of the two integrals (6) or (7). We thus can apply the integral (6) to the pseudo-current field $\mathbf{j}_1$ in order to get the magnetic field caused by the physical current density $\mathbf{j}$. This is illustrated in Figure 1 for the case of the first mode of the dynamo model of Krause and Steenbeck (1967); cf. Section 3, Equation (26) for the specification of this model.

Returning to the dynamo problem (1)–(2), we restrict ourselves to steady fields and suppose that the electric conductivity is constant, $\sigma{=}\sigma_{\mathrm{ext}}{>}0$ everywhere (see the discussion below). We insert Ohm's law (2), into the integral (6) and, exploiting the fact that $\sigma\mathbf{E} = -\sigma\,\mathrm{grad}\,\Phi$ is a gradient (for this step, the homogeneous conductivity is crucial), we can omit the term containing the electric field $\mathbf{E}$. In dimensionless variables as mentioned in the Introduction, we thus obtain

$$\boxed{\left(\underline{\underline{\hat{A}}}\mathbf{B}\right)(\mathbf{x}) := \int\limits_{\mathcal{D}} \frac{(\mathbf{x}-\mathbf{x}')\times\boldsymbol{\mathcal{F}}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\,dx'^3 = -\frac{4\pi}{C}\mathbf{B}(\mathbf{x})} \tag{8}$$

with $\boldsymbol{\mathcal{F}}$ given by Equation (3).

Equation (8) was first given by Roberts (1967, 1994) for laminar dynamos with homogeneous conductivity ($\underline{\underline{\alpha}}\equiv\underline{\underline{0}}$, $\beta\equiv 0$). It is an integral-eigenvalue equation in $\mathbf{B}(\mathbf{x})$; the spectrum of eigenvalues $\{-4\pi/C\}$ gives the critical dynamo numbers $C$, and the eigenfunctions $\mathbf{B}(\mathbf{x})$ represent the corresponding eigenmodes of the magnetic field.

In the nonlinear case, when $\mathbf{u}$, $\underline{\underline{\alpha}}$ or $\beta$ depend on $\mathbf{B}$, Equation (8) is a nonlinear integral-eigenvalue problem, which introduces some complications (but not principal ones) into the mathematical and numerical treatment.

*Thereby, in addition to (7), a surface integral $\int\limits_{\partial\mathcal{V}} \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\times d\mathbf{f}'$ occurs. But as the domain of integration in (6) and (7) is the whole space, $\mathcal{V}{=}\mathbb{R}^3$, this integral vanishes for any square integrable current field.

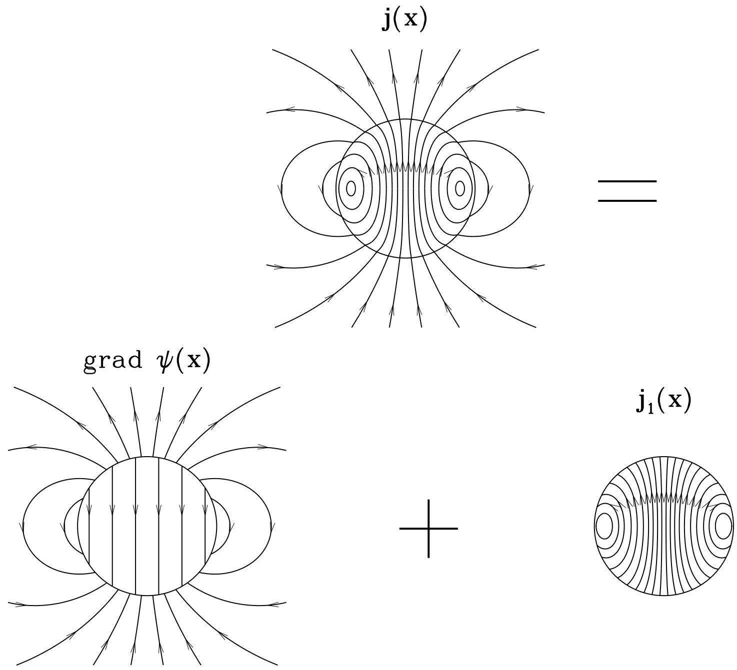


**Figure 1:** Illustration of the reduction that led to Formula (8), using the dynamo model of Krause and Steenbeck (1967). The total current field $\mathbf{j}$ is split up into a gradient, $\operatorname{grad}\psi$, and a vector field $\mathbf{j}_1$ with finite support. As the gradient does not contribute to the integral (6), the latter is reduced to an integral over the dynamo region $\mathcal{D}$ (the unit sphere) only.

It is very important that the integration in Equation (8) is not over the whole space, but only over the compact dynamo region

$$\mathcal{D} = \overline{\{\mathbf{x}|\ \underline{\underline{\alpha}}(\mathbf{x}) \neq \underline{\underline{0}}\ \vee\ \mathbf{u}(\mathbf{x}) \neq \mathbf{0}\ \vee\ \beta(\mathbf{x}) \neq 0\}}\ . \tag{9}$$

This relieves us of deep mathematical and numerical problems connected with unbounded integral operators.

Our assumption that $\sigma \equiv const$ can easily be dropped, since any variation in the conductivity $\sigma$ can be formally ascribed to the diffusivity $\beta$. Let us suppose that $\sigma$ has the copnstant value $\sigma_{\text{ext}} > 0$ outside $\mathcal{D}$ and that electrical conductivity and turbulent diffusivity within the dynamo region are given by $\tilde{\sigma}$ and $\tilde{\beta}$, respectively. Then the simple transformation

$$\sigma \quad := \quad \sigma_{\text{ext}} \tag{10}$$

$$\beta(\mathbf{x}) \quad := \quad \tilde{\beta}(\mathbf{x}) + \frac{1}{\mu_0}\left(\frac{1}{\tilde{\sigma}(\mathbf{x})} - \frac{1}{\sigma_{\text{ext}}}\right)\ . \tag{11}$$

gives us again a *formally* homogeneous conductivity $\sigma$ and a diffusivity $\beta(\mathbf{x})$ with finite support. Hence, Equation (8) is applicable whenever $\sigma$ is constant (but $\neq 0$) outside some bounded region.

In the case $\beta{\equiv}0$, the integral equation takes the form

$$\int\limits_{\mathcal{D}} \frac{(\mathbf{x}-\mathbf{x}')\times[\mathbf{u}(\mathbf{x}')\times\mathbf{B}(\mathbf{x}') + \underline{\underline{\alpha}}(\mathbf{x}')\mathbf{B}(\mathbf{x}')]}{|\mathbf{x}-\mathbf{x}'|^3}\, dx'^3 = -\frac{4\pi}{C}\mathbf{B}(\mathbf{x}) \tag{12}$$

and the operator $\underline{\underline{\hat{A}}}$ from (8) is an integral operator with *weak singularity* at $\mathbf{x}'{=}\mathbf{x}$ (we have $(\mathbf{x}-\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|^3 = \mathcal{O}(1/|\mathbf{x}-\mathbf{x}'|^2)$, thus the integral (8) exists in the strict sense). Therefore, and because the domain of integration is bounded, $\underline{\underline{\hat{A}}}$ is bounded and, moreover, *compact* (cf. Kress, 1989, theorem 2.21). Hence, Riesz' first theorem (theorem 3.1 or particularly 3.11 in the book of Kress) tells us immediately that the spectrum of $\underline{\underline{\hat{A}}}$ is countable (i. e. discrete) and has no other point of accumulation than 0 (corresponding to $C{=}\infty$). Since the integral operator $\underline{\underline{\hat{A}}}$ is not self-adjoint, there is no guarantee that the spectrum contains values other than 0. We are of course interested only in the cases in which eigenvalues other than zero exist and will discuss only them.

## 2.2 The far field

One result that can be easily derived from the integral equation (8) is the asymptotic behaviour of the *steady* field $\mathbf{B}$ far from the dynamo region:

$$\begin{aligned} \mathbf{B}(\mathbf{x}) &= -\frac{\mu_0\sigma}{4\pi}\frac{\mathbf{x}}{|\mathbf{x}|^3}\times\int \boldsymbol{\mathcal{F}}(\mathbf{x}')\, dx'^3 + \mathcal{O}(\frac{1}{r^3}) && (13)\\ &= \frac{\mu_0}{4\pi}L|\mathbf{I}_1|\frac{\sin\theta}{r^2}\mathbf{e}_\varphi + \mathcal{O}(\frac{1}{r^3}) \qquad \text{for } r = |\mathbf{x}| \to \infty && (14) \end{aligned}$$

(in dimensional form), where $(r, \theta, \varphi)$ represent polar coordinates whose $z$-axis is chosen parallel to the vector $L\mathbf{I}_1 := \sigma \int \boldsymbol{\mathcal{F}}(\mathbf{x}')\, dx'^3$. This formula is valid not only in the linear case, but rather for an arbitrary induced electromotive force $\boldsymbol{\mathcal{F}}$ of finite support. Moreover, it applies even to dynamos with varying conductivity $\sigma(\mathbf{x})$, supposed that $\lim\limits_{|\mathbf{x}|\to\infty} \sigma(\mathbf{x}){=}\sigma_\infty{>}0$ exists. In Section 4.2 we will show that for time-dependent modes the fields decay exponentially with $r$, i. e. much faster.

When the dynamo region is embedded into vacuum, the external field allows for multipole expansion and the far-field condition reads that $\mathbf{B}(\mathbf{x})$ vanishes far from the dynamo region like a dipole field ($\sim 1/r^3$); this result could also be obtained from Equation (14) in the limit $\sigma{\to}0$, $\beta{\cdot}\sigma$ fixed. In the case of a conducting cosmos however, outside the dynamo region there will in general still be a current, driven by the electric field $\mathbf{E}$ whose leading term is an electric dipole field: $\mathbf{j} \sim \mathbf{E}_{\text{dipole}} \sim 1/r^3$. This current leads to a magnetic field $\mathbf{B} \sim 1/r^2$, but as this component of $\mathbf{B}$ is merely azimuthal, it does not contradict the condition that current and magnetic flux must be localised. Roberts (1967, 1994) mentioned this far-field behaviour already in 1967 and it is not difficult to derive the asymptotics (14) from the discussion of the far-field by Meinel (1989) or, for a concrete example, from the results of Krause and Steenbeck (1967).

For most cosmic dynamos, $\underline{\underline{\alpha}}(\mathbf{x})$ is antisymmetric and $\mathbf{u}(\mathbf{x})$ and $\beta(\mathbf{x})$ are symmetric with respect to the equatorial plane (cf. Dobler and Rädler, 1998, for a detailed discussion). Then the field modes appear as symmetric fields $\mathbf{B}^{\mathcal{S}}$ or antisymmetric fields $\mathbf{B}^{\mathcal{A}}$. In the first case, $\boldsymbol{\mathcal{F}}$ is antisymmetric and $\mathbf{I}_1$ will in general be different from zero, yielding $|\mathbf{B}^{\mathcal{S}}(\mathbf{x})|{\sim}1/r^2$ for $r{\to}\infty$. However, in the second case $\boldsymbol{\mathcal{F}}$ is symmetric and $\mathbf{I}_1$ vanishes, which implies that $|\mathbf{B}^{\mathcal{A}}(\mathbf{x})|{\sim}1/r^3$. Thus, ironically, the first quadrupole-like mode generally decays slower with respect to $r$ than the first dipole-like mode. In this light, the use of the terms "quadrupole-like / dipole-like" for antisymmetric and symmetric fields seems quite questionable when the fields do not allow for multipole expansion.

## 2.3 The integral equation in cylindrical coordinates

In order to apply the integral equation (8) to concrete dynamo models, we adopt cylindrical coordinates $(\varrho, \varphi, z)$ and denote the corresponding unit vectors by $\mathbf{e}_\varrho$, $\mathbf{e}_\varphi$, $\mathbf{e}_z$. It would not be difficult to rewrite the following relations in, say, spherical coordinates. We again suppose homogeneous conductivity, $\sigma{\equiv}const$ and $\beta{\equiv}0$, since the term $\beta$-term in (8) would pose additional difficulties. For the sake of simplicity, we assume the $\alpha$-effect to be isotropic, $\alpha_{ik}(\mathbf{x}) = \alpha(\mathbf{x})\delta_{ik}$ with a scalar function $\alpha(\mathbf{x})$. Furthermore, let $\mathbf{u}$ and $\alpha$ be axisymmetric, that is, $u_\varrho$, $u_\varphi$, $u_z$ and $\alpha$ be independent of $\varphi$. Then, we can restrict ourselves to fields $\mathbf{B}(\mathbf{x}){=}B_\varrho\mathbf{e}_\varrho{+}B_\varphi\mathbf{e}_\varphi{+}B_z\mathbf{e}_z$ with $B_{\varrho/\varphi/z}(\mathbf{x}){=}\tilde{B}_{\varrho/\varphi/z}(\varrho, z)\exp(im\varphi)$ and find that the modes to different azimuthal wave numbers $m$ are not coupled.

Inserting this into the integral-eigenvalue equation (12) and carrying out the integration over azimuth $\varphi'$, we get after some algebra

$$\hat{\mathbf{I}}\,\tilde{\mathbf{B}} = -\frac{4\pi}{C}\tilde{\mathbf{B}} \tag{15}$$

for the column vector

$$\tilde{\mathbf{B}} = \begin{pmatrix} \tilde{B}_\varrho \\ \tilde{B}_\varphi \\ \tilde{B}_z \end{pmatrix}, \tag{16}$$

with the integral operator

$$\hat{\mathbf{I}} = \hat{\mathbf{I}}_{u_\varrho} + \hat{\mathbf{I}}_{u_\varphi} + \hat{\mathbf{I}}_{u_z} + \hat{\mathbf{I}}_\alpha \tag{17}$$

$$\hat{\mathbf{I}}_{u_\varrho}\tilde{\mathbf{B}} \;=\; \int dz'\,d\varrho'\,\varrho' u_\varrho(\varrho', z') \begin{pmatrix} -\varrho' E_s^m \tilde{B}_\varphi & +(z{-}z')E_c^m \tilde{B}_z \\ -(\varrho E_1^m{-}\varrho' E_c^m)\tilde{B}_\varphi & +(z{-}z')E_s^m \tilde{B}_z \\ & -(\varrho E_c^m - \varrho' E_1^m)\tilde{B}_z \end{pmatrix} \tag{18. a}$$

$$\hat{\mathbf{I}}_{u_\varphi}\tilde{\mathbf{B}} = \int dz'\, d\varrho'\, \varrho' u_\varphi(\varrho',z') \begin{pmatrix} \varrho' E_s^m \tilde{B}_\varrho & -(z-z')E_s^m \tilde{B}_z \\ (\varrho E_1^m - \varrho' E_c^m)\tilde{B}_\varrho & +(z-z')E_c^m \tilde{B}_z \\ & \varrho E_s^m \tilde{B}_z \end{pmatrix} \tag{18. b}$$

$$\hat{\mathbf{I}}_{u_z}\tilde{\mathbf{B}} = \int dz'\, d\varrho'\, \varrho' u_z(\varrho',z') \begin{pmatrix} -(z-z')E_c^m \tilde{B}_\varrho & +(z-z')E_s^m \tilde{B}_\varphi \\ -(z-z')E_s^m \tilde{B}_\varrho & -(z-z')E_c^m \tilde{B}_\varphi \\ (\varrho E_c^m - \varrho' E_1^m)\tilde{B}_\varrho & -\varrho E_s^m \tilde{B}_\varphi \end{pmatrix} \tag{18. c}$$

$$\hat{\mathbf{I}}_{\alpha}\tilde{\mathbf{B}} = \int dz'\, d\varrho'\, \varrho' \alpha(\varrho',z') \begin{pmatrix} -(z-z')E_s^m \tilde{B}_\varrho & -(z-z')E_c^m \tilde{B}_\varphi & -\varrho' E_s^m \tilde{B}_z \\ (z-z')E_c^m \tilde{B}_\varrho & -(z-z')E_s^m \tilde{B}_\varphi & -(\varrho E_1^m - \varrho' E_c^m)\tilde{B}_z \\ \varrho E_s^m \tilde{B}_\varrho & +(\varrho E_c^m - \varrho' E_1^m)\tilde{B}_\varphi & \end{pmatrix} . \tag{18. d}$$

Here $\tilde{B}_{\varrho/\varphi/z}$ stands for $\tilde{B}_{\varrho/\varphi/z}(\varrho',z')$ and $E^m_{1/c/s}$ for $E^m_{1/c/s}(\varrho,\varrho',z-z')$. The functions $E^m_{1/c/s}$ are integrals over $\varphi'$ and can be expressed in terms of hypergeometric functions; see Appendix A.1 for details.

For the axisymmetric modes of an $\alpha^2\omega$-dynamo (i. e. $m{=}0$, $\mathbf{u}{=}\omega\varrho\,\mathbf{e}_\varphi$) Equation (15) takes the form

$$-\frac{4\pi}{C}B_\varrho = -\hat{A}B_\varphi \tag{19. a}$$

$$-\frac{4\pi}{C}B_\varphi = (\hat{A}+\hat{F})B_\varrho + (\hat{D}+\hat{G})B_z \tag{19. b}$$

$$-\frac{4\pi}{C}B_z = \hat{E}B_\varphi \tag{19. c}$$

with the integral operators

$$(\hat{A}\psi)(\varrho,z) = \int dz'\, d\varrho'\, \varrho' \alpha(\varrho',z')(z-z')E_c^0 \psi(\varrho',z') \tag{20}$$

$$(\hat{D}\psi)(\varrho,z) = -\int dz'\, d\varrho'\, \varrho' \alpha(\varrho',z')\Big(\varrho E_1^0 - \varrho' E_c^0\Big)\psi(\varrho',z') \tag{21}$$

$$(\hat{E}\psi)(\varrho,z) = \int dz'\, d\varrho'\, \varrho' \alpha(\varrho',z')\Big(\varrho E_c^0 - \varrho' E_1^0\Big)\psi(\varrho',z') \tag{22}$$

$$(\hat{F}\psi)(\varrho,z) = \int dz'\, d\varrho'\, \varrho'^2 \omega(\varrho',z')(\varrho E_1^0 - \varrho' E_c^0)\psi(\varrho',z') \tag{23}$$

$$(\hat{G}\psi)(\varrho,z) = \int dz'\, d\varrho'\, \varrho'^2 \omega(\varrho',z')(z-z')E_c^0 \psi(\varrho',z') \tag{24}$$

(again the arguments of $E^0_{1/c}$ have been omitted for brevity). Equations (19) reflect the fact that differential rotation (the operators $\hat{F}$, $\hat{G}$) generates only toroidal field from poloidal one, whereas the $\alpha$-effect (operators $\hat{A}$, $\hat{D}$, $\hat{E}$) can moreover generate poloidal field from toroidal one.

From the system (19) we can eliminate $B_\varrho$ and $B_z$ and get an integral equation in $B_\varphi$ alone,

$$\Big[(\hat{D}+\hat{G})\hat{E} - (\hat{A}+\hat{F})\hat{A}\Big]B_\varphi = \left(\frac{4\pi}{C}\right)^2 B_\varphi\, . \tag{25}$$

This kind of reduction is possible only for $m{=}0$. Equation (25) is the integral equation that we solved numerically in order to get the results shown in Section 3.

# 3 Numerical results

In this section, we present some results obtained by discretising Equation (25), which holds for *steady, axisymmetric modes* of $\alpha^2\omega$-dynamos with *homogeneous conductivity*. The resulting matrix-eigenvalue problem has been solved numerically by standard techniques.

Our main purpose is to illustrate the application of our technique to concrete examples and to motivate an interpretation of some features obtained, like complex eigenvalues of the integral equation or the degeneration of dipole and quadrupole modes.

## 3.1 Spherical models

In order to verify our algorithm, we have applied it to the simple spherical $\alpha^2$-dynamo model of Krause and Steenbeck (1967), consisting of a sphere of radius $R$ with constant $\alpha$-effect:

$$\alpha(\mathbf{x}) = \begin{cases} \alpha_0 \; , & |\mathbf{x}| < R \\ 0 \; , & |\mathbf{x}| > R \end{cases} \qquad\qquad \omega \equiv 0 \; . \tag{26}$$

This model can be treated analytically.

In Table 1, the exact values for the lowest critical dynamo numbers (bottom line) are compared with numerical values for different grid sizes. The lowest critical dynamo number is $C = \mu_0\sigma\alpha_0 R \approx 3.506$, which can be compared to the estimate (46) given in Section 4.3 that yields $C{\geq}1$ in this case.

Table 1 shows that a rough picture of the distribution of critical eigenvalues is already obtained for quite coarse grids. For the finest grid applied here, the four leading dynamo numbers have a relative error less than 1.5%.

The error does not drop very quickly with grid refinement. Actually, it can be expected to decay like $\Delta C = C_N - C_{\mathrm{exact}} \sim \ln N/N$, while computation time (the number of floating point operations) is dominated by the matrix-eigenvalue algorithm, which is an $N^3$-process: $t_{\mathrm{comp}}{\sim}N^3$ for the full matrices involved here. The order $\ln N/N$ is

[†]For the nomenclature 'dipole – quadrupole', we refer only to the poloidal field here. Since in the given example $\alpha$ is symmetric with respect to the equatorial plane, the symmetry properties of poloidal and toroidal fields are opposite, not equal as in the more physical case of antisymmetric $\alpha$. For a more rigorous definition of dipole and quadrupole modes in the case of these two kinds of symmetry, cf. Dobler and Rädler, 1998.

**Table 1:** Comparison of numerical values of the first six critical dynamo numbers for different grid sizes, with the exact result for the Krause-Steenbeck dynamo (26). $N$ is the number of grid points within the dynamo region (here: within the sphere). The numbers $n$ in brackets in the last line are the mode numbers: $n = 1, 3, 5, \dots$ gives modes of dipole† symmetry, $n = 2, 4, 6, \dots$ modes of quadrupole† symmetry.

| $N$ | | | $C$ | | | |
|---|---|---|---|---|---|---|
| 26 | 3.6444 | 5.4534 | 7.2260 | 7.5984 | 9.1976 | 10.3506 |
| 40 | 3.6088 | 5.3003 | 6.9385 | 7.1936 | 8.7081 | 9.4937 |
| 120 | 3.5374 | 5.1078 | 6.5133 | 6.7236 | 7.9723 | 8.5865 |
| 196 | 3.5535 | 5.0863 | 6.4838 | 6.6915 | 7.8232 | 8.4360 |
| 394 | 3.5286 | 5.0348 | 6.3980 | 6.5933 | 7.7234 | 8.2577 |
| Theory: | 3.5059(1) | 4.9819(2) | 6.3090(3) | 6.5024(1) | 7.5651(4) | 8.0838(2) |

due to the very unsophisticated discretisation we adopted, simply replacing the integrals (20–24) over a cell $\boxed{\mathrm{i}} := [\varrho_i - \delta\varrho/2, \varrho_i + \delta\varrho/2] \times [z_i - \delta z/2, z_i + \delta z/2]$ by the value at $(\varrho_i, z_i)$, multiplied by $\delta\varrho\,\delta z$. According to the weak singularity in (8), one has to handle the case $(\varrho, z) = (\varrho_i, z_i)$, $(\varrho', z') \in \boxed{\mathrm{i}}$ separately. The simplest procedure is to drop this contribution, which introduces an error of order

$$\int_{\boxed{\mathrm{i}}} d\varrho' dz' \Big[\varrho_i E_1^0(\varrho_i, \varrho', z_i - z') - \varrho' E_c^0(\varrho_i, \varrho', z_i - z')\Big] \sim \delta\varrho\,\delta z\,\ln\delta\varrho \sim \frac{1}{N}\ln N \tag{27}$$

due to $\hat{D}$; the operators $\hat{E}$, $\hat{F}$ give terms of the same order, while the integrands in $\hat{A}$ and $\hat{G}$ are not singular at all.

We used this simple scheme because it is also the most flexible one and does not require any special treatment of points near the surface of the dynamo region. Much more effective schemes could be implemented using higher order integration formulae, but then the points close to the boundary need special treatment (as they do in higher-order finite-difference schemes).

This numerical cost of our technique is difficult to compare with finite difference schemes where only sparse matrices are involved. Firstly, the latter are often used to solve the dynamo Cauchy problem which can yield only the dynamo number and growth rate of the fastest growing mode, but never give an overview over the whole spectrum of eigenmodes and dynamo numbers. Secondly, the cost of embedding the dynamo region into a much larger volume is difficult to estimate in a general way. Finite-difference methods will be more efficient when only the leading field mode is of interest, but high accuracy is needed, since they deal with sparse matrices only, where our method involves full matrices. On the other hand, our method probably has a lead for not too high accuracy required.

The next dynamo model we examined is more physical than that of Krause and Steenbeck in assuming the $\alpha$-effect to be antisymmetric with respect to the equatorial plane:

$$\alpha(\mathbf{x}) = \begin{cases} \alpha_0 \cos\theta\ , & r < R \\ 0\ , & r > R \end{cases} \qquad\qquad \omega \equiv 0\ , \tag{28}$$

where $\alpha_0$ is constant and $\cos\theta = z/\sqrt{\varrho^2+z^2}$ denotes the cosine of the polar distance angle.

For surrounding vacuum, Roberts (1972) found the first two modes to be a dipole ($C_{\mathrm{crit}} = 7.641$) and a quadrupole ($C_{\mathrm{crit}} = 7.808$) one. Our results for homogeneous conductivity differ by about 10 % from this and are shown in Table 2. Here the first two, identical, dynamo numbers correspond to one dipole and one quadrupole mode, which are shown in Figure 2. More generally, *all* eigenmodes appear in dipole-quadrupole pairs of equal critical dynamo number, a phenomenon we will refer to as *dipole-quadrupole degeneration*.

**Table 2:** Critical dynamo numbers for the dynamo (28), showing dipole-quadrupole degeneration.

| $N$ | $C$ | | | | | |
|---|---|---|---|---|---|---|
| 394 | 6.73345 | 6.73345 | 10.5839 | 10.5839 | 11.3624 | 11.3624 |

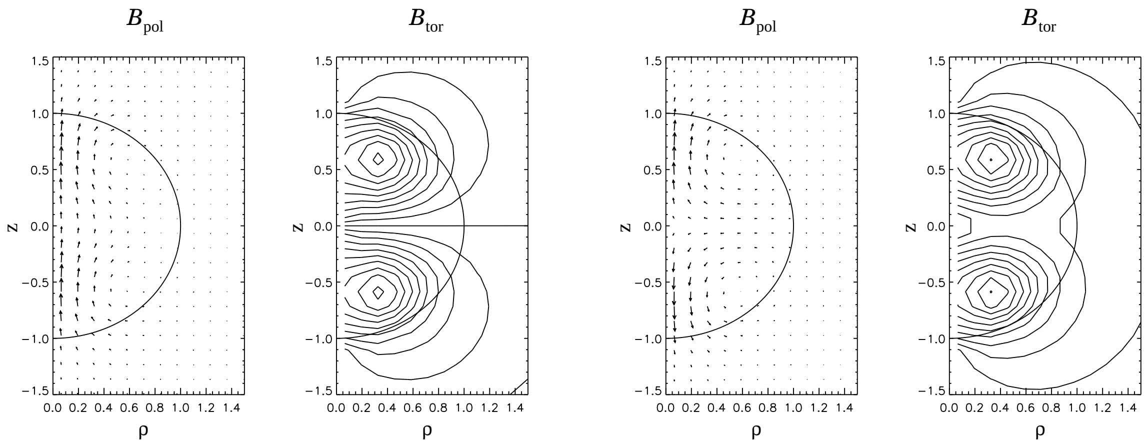


**Figure 2:** First dipole (left) and quadrupole mode (right) of the dynamo (28). Both modes correspond to a critical dynamo number $C = 6.733$.

This degeneration is related to Roberts' (1960) adjointness theorem and has been proven by Proctor (1977b, 1977c). It is not restricted to the special dynamo model

(28), but rather appears in a broad class of kinematic dynamo systems with homogeneous conductivity. This topic will be discussed in a subsequent paper (Dobler and Rädler, 1998). It should be noted that, with our numerical procedure, dipole-quadrupole degeneration is found already for arbitrarily coarse grids, which is due to the close relation of dipole-quadrupole degeneration to the integral equation (12).

## 3.2 Elliptical models

We have examined an $\alpha^2\omega$-dynamo in an "oblate" spheroid

$$\frac{\varrho^2}{a^2} + \frac{z^2}{b^2} < 1 \tag{29}$$

for the two different aspect ratios $a/b \in \{1, 3\}$ (the first one actually representing a sphere).

For $\alpha(\mathbf{x})$ and $\omega(\mathbf{x})$ we chose

$$\alpha(\mathbf{x}) = \begin{cases} \alpha_0 \dfrac{z}{b}\,, & \dfrac{\varrho^2}{a^2} + \dfrac{z^2}{b^2} < 1 \\ 0\,, & \text{otherwise} \end{cases} \tag{30}$$

$$\omega(\mathbf{x}) = \frac{C_\omega}{C_\alpha}\frac{\alpha_0}{a^2} \cdot \begin{cases} (\varrho - a)\,, & \dfrac{\varrho^2}{a^2} + \dfrac{z^2}{b^2} < 1 \\ \text{linear (in } r\text{) to zero}\,, & 1 < \dfrac{\varrho^2}{a^2} + \dfrac{z^2}{b^2} < 4 \\ 0\,, & \text{otherwise} \end{cases} \tag{31}$$

where the coefficients are chosen such that

$$C_\alpha = \mu_0 \sigma |\alpha_{\max}| a = \mu_0 \sigma \alpha_0 a\,, \qquad |C_\omega| = \mu_0 \sigma \left|\frac{\partial \omega}{\partial \varrho}\right|_{\max} a^3\,. \tag{32}$$

The interpolation of $\omega$ from the value on the surface of the ellipsoid to value zero on the surface of an embedding ellipsoid [second line in Equation (31)] was applied in order to avoid discontinuities in $\omega(\mathbf{x})$ at the surface of the (inner) ellipsoid — a problem that never arises when the dynamo is surrounded by vacuum. There are, however, no really good reasons to proceed like this as long as $\alpha(\mathbf{x})$ is still allowed to be discontinuous, because $\alpha$ and $\omega$ enter the induction equation (4) at the same level of differentiation. Thus, a discontinuity in $\omega$ should not be more problematic than one in $\alpha$, which is inherent to many dynamo models.

For both of the ellipsoids, three different ratios $C_\omega/C_\alpha \in \{0, \pm 1\}$ were examined. For $C_\omega = 0$, the $\alpha^2$-dynamo, we again have dipole-quadrupole degeneration, i. e. dipole and quadrupole modes have equal conditions of excitation, as can be seen in Table 3.

**Table 3:** Critical dynamo numbers for the $\alpha^2$-dynamo (30), $C_\omega = 0$ with different aspect ratios $a/b$.

| $\frac{a}{b}$ | $N$ | $C_\alpha$ | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 366 | 9.54674 | 9.54674 | 14.3614 | 14.3614 | 18.2074 | 18.2074 |
| 3 | 344 | 18.2931 | 18.2931 | 25.3462 | 25.3462 | 32.3976 $\pm 0.34277i$ | 32.3976 $\pm 0.34277i$ |

Table 4 shows the critical dynamo numbers for $C_\omega/C_\alpha = \pm 1$. Now, differential rotation breaks dipole-quadrupole degeneration. However, we find that changnig the sign of $C_\omega$ does not change the critical dynamo numbers, only the dipole and quadrupole modes exchange their rôles. We will call this *generalised dipole-quadrupole degeneration* and again refer to Dobler and Rädler (1998) for a detailed discussion.

Generalised dipole-quadrupole degeneration makes Table 4 valid for positive as well as negative sign of $C_\omega/C_\alpha$. For $C_\omega = +C_\alpha$, the leading mode is of dipole type, for $C_\omega = -C_\alpha$ it is of quadrupole type; both of them are shown in Figures 3 and 4.

**Table 4:** Critical dynamo numbers for the $\alpha^2\omega$-dynamo (30), (31), $C_\omega/C_\alpha = \pm 1$ (i. e. $\omega = \pm(\varrho - a)$) with different aspect ratios $a/b$.

| $\frac{a}{b}$ | $N$ | $C_\alpha$ | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 366 | 9.38634 | 9.73792 | 14.0924 | 14.6648 | 18.1471 | 18.3397 |
| 3 | 344 | 17.4321 | 19.1533 | 24.3266 | 26.7715 | 31.5100 | 32.6088 |

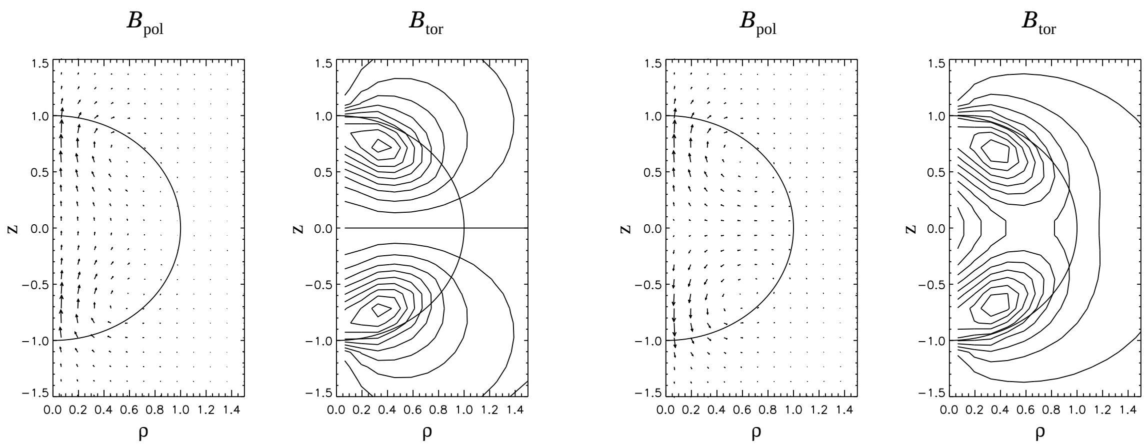


**Figure 3:** The first mode for the "ellipsoid" dynamo (30), (31) for $a$=$b$=1. Left half: $C_\omega/C_\alpha = +1$; right half $C_\omega/C_\alpha = -1$. Both modes have a critical dynamo number $C_\alpha = 9.386$.

The complex dynamo numbers in Table 3 are not physically meaningful and hence do not represent stationary field modes. However, it will become clear in

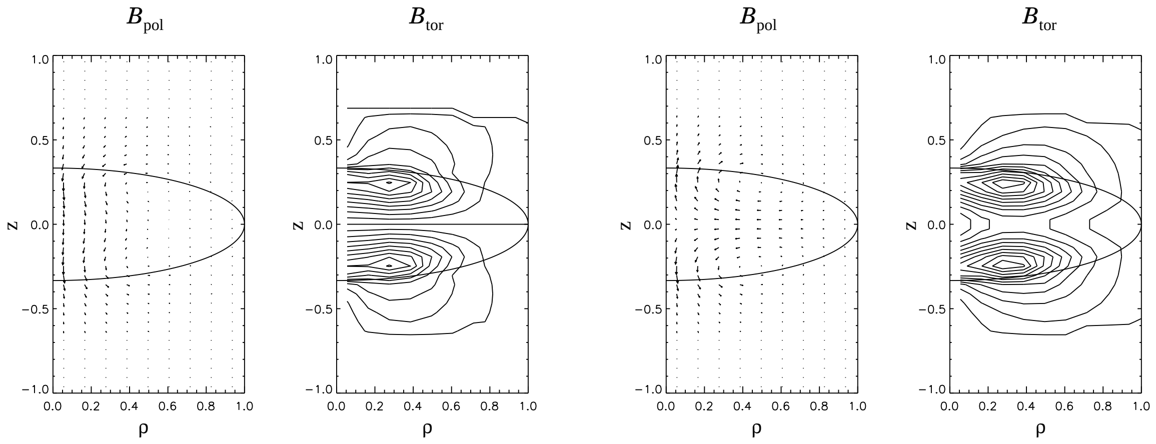


**Figure 4:** Same as Figure 3, but for $a\!=\!1$, $b\!=\!1/3$; $C_\alpha = 17.432$.

Section 4.2 that oscillating modes would appear with complex $C$ in the kind of calculations carried our here. Thus, some of the complex dynamo numbers, but not necessarily all of them, represent the oscillating modes of the system. In any case, the ordering in Table 3 is arbitrary as for the complex dynamo numbers, and only the procedure outlined in Section 4.2 will clarify their position.

Similar to what we found for spherical dynamos, one could hope that for flat spheroids ($a/b\gg1$) our results for homogeneous conductivity were in rough, qualitative agreement with calculations for thin-disc dynamos surrounded by vacuum. However, even for an aspect ratio $a/b\!=\!8$ (a model not further detailed here), we found considerable differences to results from classical thin-disc theory (cf. Ruzmaikin *et al*, 1988). There is even a strict argument showing that we can not expect to reproduce the strongly asymmetric behaviour found for the thin disc in vacuum (first quadrupole mode for $D := C_\alpha C_\omega{\cdot}b^3/a^3 \simeq -12.56$, first ["forgotten"] dipole mode for $D\!\simeq\!69.10$, cf. Soward, 1992). This is because the asymmetry is incompatible with generalised dipole-quadrupole degeneration and may therefore appear only for non-homogeneous conductivity. In other words, the qualitative features of disc dynamo models depend crucially on the conductivity outside the dynamo region — a parameter that is not always well known in astrophysical applications.

# 4 Generalisations and mathematical foundation

## 4.1 Discrete and continuous spectra

In contrast to a matrix, a linear operator on a functional space can possess an uncountable spectrum, in particular the spectrum can consist of a continuous and a discrete part. For the hermitian operators from quantum mechanics we know that the discrete spectrum is connected with localised eigenfunctions ("bound states"), while the generalised eigenfunctions, corresponding to points of the continuous spectrum, are not square integrable ("free states") and thus are no elements of the Hilbert space involved.

In the case of a finite dynamo system surrounded by vacuum, there exists only a discrete spectrum of field modes, mainly because outside the dynamo region the magnetic field is a potential field and therefore can be represented in the discrete base of multipole fields.

On the other hand, the spectrum of a bounded dynamo system embedded into conducting space will in general have both, discrete and continuous parts. The continuous spectrum is related to decaying modes which are well-known to form a continuum in the case of free decay ($\mathcal{F} \equiv 0$), while the discrete spectrum represents real dynamo action.

Meinel (1989) motivated that one can expect a similar behaviour as found in quantum mechanics for the (non-hermitian) differential operator $\underline{\underline{\hat{D}}}$ from the induction equation (4)

$$\underline{\underline{\hat{D}}}\mathbf{B} := \mathrm{curl}(\underline{\underline{\alpha}}\mathbf{B} + \mathbf{u}\times\mathbf{B} - \beta\,\mathrm{curl}\,\mathbf{B}) \;, \tag{33}$$

i. e. probably localised magnetic modes correspond to discrete eigenvalues (growth rates) $\gamma$, while the continuous spectrum is caused by non-localised modes. Whenever we use in the following sections cautious formulations like "we can expect" or "the continuum should be on the negative half axis", we argue on the base of this analogy and the corresponding results are not mathematically strict but only strongly motivated; this is in contrast to the strict results on boundedness and compactness for the integral operators in Equations (12) and (40) below.

As we will see in this section, the derivation of our integral equation (8) or (40) implies that the magnetic field modes $\mathbf{B}(\mathbf{x})$ are localised. This means that in the integral-equation formalism non-localised fields and, if the analogy to quantum mechanics holds, the continuous spectrum are discarded from the very start (but only for growth rates $\gamma$ that are not real negative, see below). As Meinel shows, these modes always decay with time and are thus of minor interest in dynamo theory. It is only by excluding these non-localised modes, that some severe problems connected with the differential operator from the induction equation are overcome, and an integral-equation description of the dynamo problem is possible.

## 4.2 Time dependent magnetic field

The most general integral equation for the time-dependent case (but assuming constant conductivity outside the dynamo region) is obtained by transforming the induction equation (4) into the integral equation

$$\begin{aligned}\mathbf{B}(\mathbf{x},t) &= \int\limits_{\mathrm{I\!R}^3} dx'^3 \; G(\mathbf{x}-\mathbf{x}',t)\,\mathbf{B}_0(\mathbf{x}') \\ &\quad - \int\limits_0^t dt' \int\limits_{\mathcal{D}} dx'^3 \; \mathrm{grad}'\, G(\mathbf{x}-\mathbf{x}',t-t')\times\boldsymbol{\mathcal{F}}(\mathbf{x}',t')\;. \end{aligned} \tag{34}$$

Equation (34) is a variant of an equation that can be found in an article by Rädler (1968) or the book of Krause and Rädler (1980). Here,

$$G(\mathbf{x}-\mathbf{x}',t) = \frac{1}{(4\pi\eta t)^{3/2}} e^{-\frac{(\mathbf{x}-\mathbf{x}')^2}{4\eta t}} \tag{35}$$

denotes Green's function of the heat-conduction operator on the left of Equation (4) and $\mathbf{B}_0(\mathbf{x}) := \mathbf{B}(\mathbf{x},t{=}0)$ is the initial field. $\eta{=}1/(\mu_0\sigma)$ denotes the magnetic diffusivity and like in Section 2.1, variations in $\sigma(\mathbf{x})$ are transmuted into $\beta(\mathbf{x})$. Although important as an analytical tool, Equation (34) is too general for numerical applications and we will now turn to a more particular formulation.

If we introduce the ansatz

$$\mathbf{B}(\mathbf{x},t) = \tilde{\mathbf{B}}(\mathbf{x})e^{\gamma t}, \qquad \gamma \text{ complex}\;, \tag{36}$$

into the induction equation (4) and omit the tilde, we get

$$\frac{1}{C}(\Delta\mathbf{B} - \gamma\mathbf{B}) = -\,\mathrm{curl}\,\boldsymbol{\mathcal{F}}\;, \qquad \mathrm{div}\,\mathbf{B} = 0\;, \tag{37}$$

with $\boldsymbol{\mathcal{F}}$ given by (3). For $C$ fixed, Equation (37) is an eigenvalue problem with eigenvalues $\gamma$. Formally, we can as well fix $\gamma$ and get a generalised eigenvalue problem with eigenvalues $1/C$ which will in general be complex. In analogy to the case of general time dependence, we can use Green's function $G(\mathbf{x}, xv')$ of the Helmholtz-type operator in Equation (37) to transform (37) into an integral equation:

$$\mathbf{B}(\mathbf{x}) = -C\int\limits_{\mathcal{D}} G(\mathbf{x},\mathbf{x}')\,\mathrm{curl}'\,\boldsymbol{\mathcal{F}}\,dx'^3 \;= C\int\limits_{\mathcal{D}} \mathrm{grad}'\,G(\mathbf{x},\mathbf{x}')\times\boldsymbol{\mathcal{F}}\,dx'^3\;. \tag{38}$$

For the boundary condition of square integrability, Green's function $G(\mathbf{x},\mathbf{x}')$ is well known (cf. Morse and Feshbach, 1953),

$$G(\mathbf{x},\mathbf{x}') = G(\mathbf{x}-\mathbf{x}') = -\frac{1}{4\pi}\frac{e^{-\sqrt{\gamma}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\;, \tag{39}$$

where the root $\sqrt{\gamma}$ of a complex number is chosen such that $\mathfrak{Re}\,\sqrt{\gamma}\geq 0$. Note that $G$ is only unique (and compatible with the condition of square integrability) if $\gamma$ *is not* *on the negative real axis,* $\gamma \notin \mathbb{R}^-$, and we will restrict ourselves to that case here and in the following. Inserting (39) into Equation (38), we get

$$\boxed{\left(\hat{\underline{\underline{A}}}^{(\gamma)}\mathbf{B}\right)(\mathbf{x}) = \int\limits_{\mathcal{D}} \frac{(\mathbf{x}-\mathbf{x}')\times\boldsymbol{\mathcal{F}}}{|\mathbf{x}-\mathbf{x}'|^3}\, e^{-\sqrt{\gamma}|\mathbf{x}-\mathbf{x}'|}\left(1+\sqrt{\gamma}|\mathbf{x}-\mathbf{x}'|\right)dx'^3 = -\frac{4\pi}{C}\mathbf{B}(\mathbf{x})} \, . \tag{40}$$

Like in Section 2.1 it is easily shown that the operator on the left hand side of (40) is compact in the case of homogeneous conductivity (and thus bounded) for $\gamma \notin \mathbb{R}^-$.

For a given value of $\gamma$, Equation (40) is again an integral-eigenvalue equation for $\mathbf{B}(\mathbf{x})$ with eigenvalues $-4\pi/C$. In order to get numerically the modes with a given growth rate $\mathfrak{Re}\,\gamma$ and the corresponding dynamo numbers (e. g. $\mathfrak{Re}\,\gamma = 0$ and $C = C_{\mathrm{crit}}$) one has to solve Equation (40) numerically for different values of $\Omega := \mathfrak{Im}\,\gamma$, which yields complex functions $C(\Omega)$ that are continuous because they are eigenvalues of an integral operator that depends smoothly on $\gamma$ (and so does the matrix obtained by simple discretisation). One has to find the zeros of the imaginary part of these functions, $\mathfrak{Im}\,C(\Omega) \overset{!}{=} 0$, because only real dynamo numbers are physically meaningful. The corresponding value of $\Omega$ is then the oscillation frequency of the time dependent mode.

For $\gamma \neq 0$, $\gamma \notin \mathbb{R}^-$, we have $\mathfrak{Re}\,\sqrt{\gamma} > 0$, and $\mathbf{B}(\mathbf{x})$ decays for $r := |\mathbf{x}| \to \infty$ exponentially according to

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_0\sigma}{4\pi} e^{-\sqrt{\frac{\gamma}{\eta}}\,r} \sqrt{\frac{\gamma}{\eta}} \left(\frac{1}{r} + \mathcal{O}(\frac{1}{r^2})\right) \frac{\mathbf{x}}{r} \times L\mathbf{I}_1 \tag{41}$$

with $L\mathbf{I}_1 = \sigma \int \boldsymbol{\mathcal{F}}(\mathbf{x}')\, dx'^3$.

In an infinite conductor, the equation $\eta\Delta\mathbf{B} - \gamma\mathbf{B} = \mathbf{0}$ for the free decay of magnetic field has a continuous spectrum $\gamma = -\eta\mathbf{k}^2$, $\mathbf{k} \in \mathbb{R}^3$ and the corresponding eigenfunctions $\mathbf{B} \sim j_l(kr)\cdot Y_l^m(\vartheta, \varphi)$ — with $j_l$ denoting spherical Bessel functions — tend to zero for $r \to \infty$ (at least they can be chosen so), but are not localised. We can expect the dynamo equation (4) to have such a continuum, too, and since only for real and negative $\gamma$ modes with $\mathbf{B} \to \mathbf{0}$ for $r \to 0$ may be non-localised, the continuum should still be on the negative half axis.

## 4.3 A necessary condition for magnetic field generation

We already mentioned that the integral operators in Equations (8) and (40) are bounded for $\beta \equiv 0$. It is not difficult to derive a concrete upper bound for the norm $\|\hat{\underline{\underline{A}}}^{(\gamma)}\|$. Let $\|\mathbf{u}\|_\infty = \max\limits_{\mathbf{x}\in\mathcal{D}} |\mathbf{u}(\mathbf{x})|$ and $\|\underline{\underline{\alpha}}\|_\infty = \max\limits_{\mathbf{x}\in\mathcal{D}} \|\underline{\underline{\alpha}}(\mathbf{x})\|_2$ (where $\|\underline{\underline{\alpha}}\|_2$ is the spectral norm, or any other matrix norm that is compatible with the Euclidean vector norm) denote the maximum norms of the vector function $\mathbf{u}(\mathbf{x})$ and the tensor function $\underline{\underline{\alpha}}(\mathbf{x})$;

let $L_{\mathcal{D}}$ the diameter of the smallest sphere enclosing the dynamo region $\mathcal{D}$. Noting that $0 < e^{-\sqrt{\gamma}|\mathbf{x}|}\left(1+\sqrt{\gamma}|\mathbf{x}|\right) \leq 1 \quad \forall\, \mathbf{x}$, we find for an arbitrary bounded vector function $\mathbf{b}$ in the dimensionless units from (4) and (8):

$$\begin{aligned}
\left|(\underline{\underline{\hat{A}}}^{(\gamma)}\,\mathbf{b})(\mathbf{x})\right| &\leq \int_{\mathcal{D}} \left|\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right| \cdot \left|\mathbf{u}(\mathbf{x}')\times\mathbf{b}(\mathbf{x}') + \underline{\underline{\alpha}}(\mathbf{x}')\mathbf{b}(\mathbf{x}')\right| dx'^3 && (42)\\
&\leq \int_{\mathcal{D}} \frac{dx'^3}{|\mathbf{x}-\mathbf{x}'|^2} \cdot \left(\|\mathbf{u}\|_\infty + \|\underline{\underline{\alpha}}\|_\infty\right) \cdot \|\mathbf{b}\|_\infty && (43)\\
&\leq 2\pi L_{\mathcal{D}} \cdot \left(\|\mathbf{u}\|_\infty + \|\underline{\underline{\alpha}}\|_\infty\right) \cdot \|\mathbf{b}\|_\infty \;. && (44)
\end{aligned}$$

In other words, $\underline{\underline{\hat{A}}}^{(\gamma)}$ is bounded, if $\mathcal{D}$ is and $\underline{\underline{\alpha}}$ and $\mathbf{u}$ are, and then

$$\|\underline{\underline{\hat{A}}}^{(\gamma)}\| \leq 2\pi L_{\mathcal{D}}\left(\|\mathbf{u}\|_\infty + \|\underline{\underline{\alpha}}\|_\infty\right) \;. \tag{45}$$

It is essential here that our operator acts on functions on a bounded region $\mathcal{D}$, because on unbounded regions the maximum modulus $\max_{\mathbf{x}} |f(\mathbf{x})|$ of a function $f(\mathbf{x})$ is not a norm of $f$.

The same argumentation as above is valid in the case of $\alpha$-quenching, provided that the norms of $\mathbf{u}$ and $\underline{\underline{\alpha}}$ are now defined by $\|\mathbf{u}\|_\infty = \max_{\mathbf{x},\mathbf{B}} |\mathbf{u}(\mathbf{x};\mathbf{B})|$, $\|\underline{\underline{\alpha}}\|_\infty = \max_{\mathbf{x},\mathbf{B}} \|\underline{\underline{\alpha}}(\mathbf{x};\mathbf{B})\|_2$ and are finite.

As the eigenvalues of a linear operator are bounded by the operator norm (cf. Kress, 1989), we derive from the upper bound (45) as a corollary an estimate for the dynamo numbers of steady (linear or nonlinear) and time dependent (with $\gamma \notin \mathbb{R}^-$) dynamos with homogeneous conductivity ($\sigma \equiv const$):

$$\mu_0\,\sigma L_{\mathcal{D}}\left(\|\mathbf{u}\|_\infty + \|\underline{\underline{\alpha}}\|_\infty\right) \geq 2 \;. \tag{46}$$

This is a *necessary condition* for the excitation of magnetic field by mean-field dynamos with homogeneous conductivity. It has first been derived by Roberts (1967, 1994) for the steady case.

Other necessary conditions have been given by Backus (1958) and Childress (1969). The condition of Childress (1969) is similar to ours in that it gives an estimate for the magnitude of the velocity field itself. For a spherical dynamo with constant electrical conductivity and solenoidal motions, it reads

$$\mu_0\sigma R\|\mathbf{u}\|_\infty \geq \mu_0\sigma R\frac{|\Delta_{\max}\mathbf{u}|}{2} \geq \frac{\pi}{2} \;, \tag{47}$$

where $\Delta_{\max}\mathbf{u}$ is the maximum relative velocity inside the sphere. For a sphere, our estimate (46) yields the weaker result $|C_{\mathrm{crit}}| = \mu_0\sigma R\|\mathbf{u}\|_\infty \geq 1$.

For a cylinder of height $2h$ and radius $R$, the "geometric integral" in (43) can also be given explicitely and we get

$$\mu_0\,\sigma h\left(\|\mathbf{u}\|_\infty+\|\underline{\underline{\alpha}}\|_\infty\right) \geq \frac{1}{\ln\dfrac{R}{h}+1+\dfrac{h}{2R}} \;; \tag{48}$$

for a thin disc, the term $h/(2R)$ can be neglected.

As (46) holds for time dependence $\sim \exp(\gamma t)$ with $\gamma\notin\mathbb{R}^-$, we can conclude directly that for dynamo numbers $C$ below this lower bound only decay of the magnetic field modes is possible and that this decay occurs with $\gamma\in\mathbb{R}^-$, i. e. non-oscillatory, as otherwise a contradiction to the condition (46) would occur. Nothing similar can be said when the dynamo number is above the bound (46), but below the lowest critical value.

## 4.4 Variable conductivity

When the function $\beta(\mathbf{x})$ is different from zero on a positive volume, the integral operator $\underline{\underline{\hat{A}}}$ from Equation (8) is no longer compact. This is not very surprising, since we apply the integral operator "$\mathrm{curl}^{-1}$" $= \int \mathrm{grad}' \frac{1}{|\mathbf{x}-\mathbf{x}'|} \times \cdot \; dx'^3$ to $\mathrm{curl}'\,\mathbf{B}$ and it is well konwn that the identity operator is bounded, but not compact (cf. Kress, 1989). More insight can be gained by eliminating the differentiation of $\mathbf{B}$, integrating (8) by parts according to

$$\int\limits_{\mathcal{V}} \mathbf{a}\times \mathrm{curl}'\,\mathbf{b}\, dx'^3 = -\int\limits_{\partial\mathcal{V}} \mathbf{a}\times[\mathbf{b}\times d\mathbf{f}'] + \int\limits_{\mathcal{V}} (\mathbf{b}\ \mathrm{div}'\,\mathbf{a} - \underbrace{\mathbf{b}\nabla'\mathbf{a}})\, dx'^3 \;, \tag{49}$$

with $(\underbrace{\mathbf{b}\nabla'\mathbf{a}})_i := b_j\partial'_i a_j$, and using the well-known formula from potential theory

$$\mathrm{div}'\,\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -4\pi\delta(\mathbf{x}-\mathbf{x}')\;. \tag{50}$$

We thus get the following integral equation in non-dimensional form

$$\begin{aligned} \int\Bigg\{ & \frac{(\mathbf{x}-\mathbf{x}')\times[\alpha\mathbf{B}+\mathbf{u}\times\mathbf{B}+\mathrm{grad}'\,\beta\times\mathbf{B}]}{|\mathbf{x}-\mathbf{x}'|^3} \\ & + \quad \beta(\mathbf{x}')\left(\frac{\mathbf{B}}{|\mathbf{x}-\mathbf{x}'|^3} - 3\frac{(\mathbf{B}\cdot(|\mathbf{x}-\mathbf{x}'|))}{|\mathbf{x}-\mathbf{x}'|^5}(|\mathbf{x}-\mathbf{x}'|)\right)\Bigg\}\, dx'^3 \\ + \quad 4\pi\beta(\mathbf{x})\mathbf{B}(\mathbf{x}) \qquad & = \qquad -\frac{4\pi}{C}\mathbf{B}(\mathbf{x})\;. \end{aligned} \tag{51}$$

Note that the term $\mathrm{grad}'\,\beta\times\mathbf{B}$ does *not* represent turbulent diamagnetism, since the latter is included in $\mathbf{u}\times\mathbf{B}$ from the very start.

This is again a kind of eigenvalue equation for $\mathbf{B}(\mathbf{x})$ on $\mathcal{D}$ with eigenvalues $-4\pi/C$. There are, however, two facts that make its mathematical and numerical treatment much more complicated than that of Equation (8). First, the term

$\underline{\underline{\hat{A}}}_{\rm III}\mathbf{B} := 4\pi\beta(\mathbf{x})\cdot\mathbf{B}(\mathbf{x})$, like the identity operator on function spaces, is bounded, but not compact (cf. Kress, 1989, theorem 2.19). Hence, Equation (51) is no longer an eigenvalue problem for a compact operator and, deprived of the power of Riesz theory, we no longer can exclude the existence of a continuous spectrum. Second, the kernel of the second integral operator $\underline{\underline{\hat{A}}}_{\rm II}$

$$(\underline{\underline{\hat{A}}}_{\rm II}\mathbf{B})(\mathbf{x}) := \int \beta(\mathbf{x}')\left(\frac{\mathbf{B}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} - 3\frac{(\mathbf{B}(\mathbf{x}')\cdot(\mathbf{x}-\mathbf{x}'))}{|\mathbf{x}-\mathbf{x}'|^5}(\mathbf{x}-\mathbf{x}')\right) dx'^3 \tag{52}$$

is singular (not only weakly singular as above) and the integral exists only as a principal value. $\underline{\underline{\hat{A}}}_{\rm II}$ is even unbounded on spaces of continuous functions, but bounded on Hölder spaces. In any case it is not a compact operator.

Probably, an integral equation involving a Green's function for the given conductivity distribution would again give us a compact operator. But even for the simple case of a sphere of constant conductivity in vacuum, there seems to be no closed analytical expression for the corresponding Green's function and it can only be given by an infinite series in special functions (cf. Bräuer and Rädler, 1987).

Of course, these questions should be investigated in more detail. It may be possible to overcome the difficulties, but the numerical solution of (51) can prove to be much less straightforward than in the case of homogeneous conductivity.

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# Appendix

## A.1 The integrals $E^m_{1/c/s}$

In Section 2.3 we introduced three symbols $E_1^m$, $E_c^m$, $E_s^m$, that are defined as follows:

$$E_1^m = E_1^m(\varrho,\varrho',z-z') \;=\; 2\int_0^\pi \frac{\cos m\varphi'}{[\varrho^2+\varrho'^2-2\varrho\varrho'\cos\varphi'+(z-z')^2]^{3/2}}\,d\varphi' \tag{53}$$

$$E_c^m = E_c^m(\varrho,\varrho',z-z') \;=\; 2\int_0^\pi \frac{\cos\varphi'\cos m\varphi'}{[\varrho^2+\varrho'^2-2\varrho\varrho'\cos\varphi'+(z-z')^2]^{3/2}}\,d\varphi' \tag{54}$$

$$E_s^m = E_s^m(\varrho,\varrho',z-z') \;=\; 2i\int_0^\pi \frac{\sin\varphi'\sin m\varphi'}{[\varrho^2+\varrho'^2-2\varrho\varrho'\cos\varphi'+(z-z')^2]^{3/2}}\,d\varphi'\;. \tag{55}$$

These integrals can be expressed in terms of complete elliptic integrals, but as far as we see, this results in no general expressions that apply to arbitrary $m$.

Alternatively, we can express the integrals in terms of hypergeometric functions. Formula 30 in §5 of Oberhettinger's (1957) table of Fourier transforms, in connection with a limiting procedure and some standard multiplication theorems for trigonometric functions, can be used to obtain the representation

$$E_1^0(\varrho,\varrho',\zeta) \;=\; \frac{v_0}{2}\,{}_2F_1(-\tfrac{1}{2},-\tfrac{1}{2};1;e^{-2a}) \tag{56}$$

$$E_c^0(\varrho,\varrho',\zeta) \;=\; \frac{3}{4}v_0e^{-a}\,{}_2F_1(-\tfrac{1}{2},\tfrac{1}{2};2;e^{-2a}) \tag{57}$$

$$E_s^0(\varrho,\varrho',\zeta) \;=\; 0\,, \tag{58}$$

and, for $m>0$,

$$E_1^m(\varrho,\varrho',\zeta) \;=\; v_m(m+\tfrac{1}{2})\,{}_2F_1(-\tfrac{1}{2},m-\tfrac{1}{2};m+1;e^{-2a}) \tag{59}$$

$$E_c^m(\varrho,\varrho',\zeta) \;=\; \frac{v_m}{2}\Big[\,me^a\,{}_2F_1(-\tfrac{1}{2},m-\tfrac{3}{2};m;e^{-2a})\,+$$

$$+\frac{(m+\frac{1}{2})(m+\frac{3}{2})}{m+1}e^{-a}\,{}_2F_1(-\tfrac{1}{2},m+\tfrac{1}{2};m+2;e^{-2a})\Bigg] \tag{60}$$

$$\begin{aligned} E_s^m(\varrho,\varrho',\zeta) &= i\frac{v_m}{2}\Bigg[\,me^a\,{}_2F_1(-\tfrac{1}{2},m-\tfrac{3}{2};m;e^{-2a}) - \\ &\quad -\frac{(m+\frac{1}{2})(m+\frac{3}{2})}{m+1}e^{-a}\,{}_2F_1(-\tfrac{1}{2},m+\tfrac{1}{2};m+2;e^{-2a})\Bigg]\ . \end{aligned} \tag{61}$$

Here the abbreviations

$$a = \operatorname{arcosh}\frac{\varrho^2+\varrho'^2+\zeta^2}{2\varrho\varrho'} = \arcsin\frac{\sqrt{(\varrho^2-\varrho'^2)^2+2(\varrho^2+\varrho'^2)\zeta^2+\zeta^4}}{2\varrho\varrho'} \tag{62}$$

$$v_m = \frac{\pi e^{-a(m-\frac{1}{2})}}{(\varrho\varrho')^{3/2}\sinh^2 a}\cdot\frac{(2m)!}{2^{2m}(m!)^2} = \frac{\sqrt{\pi}e^{-a(m-\frac{1}{2})}}{(\varrho\varrho')^{3/2}\sinh^2 a}\cdot\frac{(m-\frac{1}{2})!}{m!} \tag{63}$$

are used, and

$${}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{x^n}{n!} \tag{64}$$

denotes Gauss' hypergeometric function (cf. Abramowitz and Stegun, 1980).