The Mean-Field Dynamo Equations with Discontinuous Coefficients and Parker's Slab Model

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September 25, 2000

Abstract

The paper discusses the properties of dynamo models with discontinuous distribution of the induction coefficients, showing that no additional source terms appear due to the discontinuities.

The apparent contradiction between Parker's (1971) result for his $\alpha\omega$-dynamo model and later results, which gave rise to the introduction of additional terms by Ruzmaikin *et al* (1979), is shown to be due to misinterpretation. Parker's fields agree well with both, the exact solution and numerical results obtained with a finite-difference scheme.

It is shown that the $\alpha\omega$-approximation is justified even for a discontinuous α-coefficient, provided that the magnetic Reynolds numbers R_ω and R_α are large and small enough, respectively.

1 Introduction

One of the difficulties connected with kinematic dynamo theory lies in the fact that the number of analytically solvable dynamo models retaining the basic physical properties of real dynamos is quite low. Such models would, for example, provide a natural link between numerical solutions that are

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easily accessible only in some vicinity of the threshold of generation, and asymptotic results valid for large magnetic Reynolds numbers.

Spherical models are solvable in closed form if the α-effect is constant (Krause & Steenbeck, 1967) or a function of radius only, e. g. piecewise constant. This contradicts the antisymmetry of $\alpha(\mathrm{x})$ with respect to the equatorial plane that is essential for most cosmic dynamo objects (cf. Krause & Rädler, 1980).

Moffatt (1978) suggested an exactly solvable slab model consisting of two planes with vertically concentrated α-effect, namely

$$\alpha(z) = \delta(z+\zeta) - \delta(z-\zeta) \; . \tag{1}$$

This profile has the right symmetry and some resemblance to galactic disc dynamos, but the strong concentration of one of the induction effects seems unphysical. Nevertheless, this kind of model has proven to be useful for discussing interesting physical effects in analytical studies (see for example Gabov *et al*, 1996). One of the main advantages of the model is that it is analytically solvable even as a full $\alpha^2\omega$-dynamo model and the solution is quite compact (cf. Ruzmaikin *et al*, 1980b).

The third type of analytically solvable dynamo models consists of slab models with an α-effect that is a piecewise constant function of the vertical coordinate z only. Typically, α is assumed constant in the upper and lower part of the disc, respectively, but changes sign at the equatorial plane,

$$\alpha(z) = \alpha_0 \operatorname{sgn} z \qquad \text{for } |z|<h \; . \tag{2}$$

Such a model of an $\alpha\omega$-dynamo was presented in a classical article by Parker (1971) and is clearly closer to physical reality that Moffatt's model. The corresponding α^2-model can also be solved and has been discussed by Ruzmaikin *et al* (1980b) and Rädler & Bräuer (1987). Meinel (1990) solved the non-axisymmetric problem for a finite α^2-cylinder, but had to suppose a perfectly conducting medium outside the disc.

Ruzmaikin *et al* (1979, 1988) apparently found Parker's results in qualitative disagreement with both, numerical results for similar distributions of the induction effects and some general relations for thin-disc dynamos. To resolve this contradiction, they introduced into the thin-disc dynamo equation an additional source term proportional to the step in α. Recently, Sokoloff (1997) gave the general form of this additional term for arbitrary surfaces of discontinuity.

This additional source term was introduced as the expression of principial changes in physics, appearing due to the discontinuity of $\alpha(\mathbf{x})$ at the midplane $z{=}0$. According to the argumentation of Ruzmaikin *et al* (1979), even models with continuous approximations to the step function,

$$\alpha(z) = \mathrm{sgn}_\varepsilon(z) \underset{\varepsilon\to 0}{\longrightarrow} \mathrm{sgn}(z) \tag{3}$$

would yield results that are principally different from the discontinuous case. Such behaviour would be very disturbing as it contradicts our physical intuition fed by a lot of examples where comparable approximations work successfully. It is the aim of the present paper to return sound sleep to researchers by showing that discontinuous distributions of the induction effects do not introduce any new physics or additional terms, and that dealing with them, one needs not be afraid of unexpected effects.

The argument that α-profiles with structures smaller than the turbulent correlation length l_{turb} are unphysical (Ruzmaikin *et al*, 1980a, 1980b) is a bit too conservative. Averaging over azimuth φ (cf. Braginskij, 1964a, 1964b, 1964c; Krause & Rädler, 1980) can of course lead to arbitrarily sharp distributions of α, and so can ensemble averaging in systems where the ergodic hypothesis (concerning space averages) is inapplicable. Apart from this, after the averaging procedure has been carried out, the remaining equations no longer contain any information about the scale l_{turb} and may be examined in their own right.

We will show that the results of Parker are in good agreement with similar models like the ones presented by Ruzmaikin *et al* (1988) and have been misinterpreted by Ruzmaikin *et al* (1979, 1988), which was facilitated by unlucky presentation. Thus, among the analytically solvable dynamo models, Parker's model is probably one of the most physical ones and provides a good tool in cases where exact solutions are needed for disc dynamos.

2 The thin disc

Let us regard the induction equation for a thin disc of half-thickness h, with an antisymmetric α-effect that is discontinuous at the equatorial plane $z = 0$:

$$\alpha = \frac{[\![\alpha]\!]}{2}\,\mathrm{sgn}\,z \qquad \text{for } |z| \ll h\ , \tag{4}$$

where the brackets $[\![\psi]\!](x,y) := \psi(x,y,+0) - \psi(x,y,-0)$ denote the jump of the quantity ψ at the equatorial plane.

In cylindrical coordinates (r,φ,z), the equation for the radial component B_r and the azimuthal component B_φ of the magnetic field in a thin disc are

$$\frac{\partial B_r}{\partial t} = -(\alpha B_\varphi)' + (\beta B_r')' \tag{5}$$

$$\frac{\partial B_\varphi}{\partial t} = (\alpha B_r)' + G B_r + (\beta B_\varphi')' \,. \tag{6}$$

Here, primes denote differentiation with respect to the vertical coordinate z. The assumption has been made that the magnetic field is axisymmetric and its radial scale is excessively larger that the disc thickness $2h$. The coefficient α represents the α-effect, $G := r\,\partial\omega/\partial r$ the shear strength and β is the turbulent diffusion coefficient. Note that, strictly speaking, variations in β will introduce an additional vertical advection term $1/2\,(\beta' B_{r/\varphi})'$ in (5), (6) due to "turbulent diamagnetism" (cf. Rädler, 1968; Vainshtein & Zeldovich, 1972; Gabov *et al*, 1996). As this effect does not alter our argumentation, we drop it for sake of simplicity.

We supplement Equations (5), (6) by the standard boundary conditions

$$B_r(\pm h) = 0 \tag{7}$$

$$B_\varphi(\pm h) = 0\,, \tag{8}$$

noting however that this excludes some odd (dipole) modes of the thin disc (Rädler & Bräuer, 1987; cf. Soward, 1992, or Dobler & Walker, 1997, for a detailed discussion).

In the applications to astrophysical discs, the term $(\alpha B_r)'$ in (6) is usually neglected since it is dominated by differential rotation GB_r, which leads to so-called $\alpha\omega$-dynamo models. For discontinuous α, this assumption seems to be problematic, because $\alpha' \sim \delta(z)$ is surely $\gg G$ in the equatorial plane. This is however a too naïve argument and we will justify the $\alpha\omega$-approximation for this case in Section 4.

2.1 Jump conditions for the magnetic field

Following Ruzmaikin *et al* (1979) and Sokoloff (1997) we integrate Equations (5), (6) across the surface of discontinuity. The important contribution comes from near the equatorial plane, and supposing B_r and B_φ to be continuous (i. e. excluding surface currents) the limit $\lim\limits_{\varepsilon\to 0}\int_{\varepsilon}^{\varepsilon}(5),(6)\,dz$ yields

$$
\begin{aligned}
0 &= -[\![\alpha]\!]B_\varphi(0) + [\![\beta B_r']\!] && (9)\\
0 &= \quad [\![\alpha]\!]B_r(0) + [\![\beta B_\varphi']\!]\ . && (10)
\end{aligned}
$$

These jump conditions must be fulfilled when matching solutions of Equations (5)–(8) on the upper with solutions on the lower half of the disc.

Condition (9) shows that for even (quadrupole) modes B_r' will be discontinuous if $[\![\alpha]\!]{\neq}0$. Particularly, instead of the relations $B_r'(0){=}0$, $B_\varphi'(0){=}0$ holding for the even modes of a dynamo with continuous α, we now have

$$
\begin{aligned}
-B_r'(-0) &= B_r'(+0) = \frac{1}{\beta(0)}\frac{[\![\alpha]\!]}{2}B_\varphi(0) && (11)\\
-B_\varphi'(-0) &= B_\varphi'(+0) = -\frac{1}{\beta(0)}\frac{[\![\alpha]\!]}{2}B_r(0)\ , && (12)
\end{aligned}
$$

supposing $\beta(z)$ to be continuous. Integrating Equations (5), (6) from 0 to h we get

$$
\begin{aligned}
\frac{d}{dt}\int_0^h B_r\,dz &= \beta(h)B_r'(h) && (13)\\
\frac{d}{dt}\int_0^h B_\varphi\,dz &= G\int_0^h B_r\,dz + \beta(h)B_\varphi'(h) && (14)
\end{aligned}
$$

for even modes. Ruzmaikin *et al* (1979) arrived at a different formula for $\int B_r\,dz$, namely Equation (13) with an additional term $-[\![\alpha]\!]B_\varphi(0)$. This is probably due to the erroneous assumption that $B_r'(\pm 0){=}0$. If this were the case, of course (9) could not be fulfilled (at least for $B_\varphi(0){\neq}0$) and one would really need an additional source term in the dynamo equation (5). But as things are, the jump in αB_φ is compensated by the one in $\beta B_r'$ and there is no need (and no justification) for complicating Equations (5), (6) by adding any terms.

2.2 An alternative approach

The jump condition (9), (10) can also be obtained in an alternative way, starting from Maxwell's equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\operatorname{curl}\mathbf{E}\ . \tag{15}$$

Excluding magnetic flux sheets that would lead to infinite magnetic energy density, Equation (15) yields that

$$[\![\mathbf{E}_{\mathrm{tang}}]\!] = \mathbf{0} \tag{16}$$

on any boundary, i. e. the tangential components $\mathbf{E}_{\mathrm{tang}}$ of the electric field are continuous[1]. Condition (16) was used in several dynamo models, cf. for example Krause & Rädler (1980).

Now Ohm's law implies that

$$\mathrm{curl}\,\mathbf{B} = \mu_0 \mathbf{j} = \frac{1}{\beta}(\mathbf{E} + \mathbf{u}\times\mathbf{B} + \alpha\mathbf{B}) \;, \tag{17}$$

that is,

$$\mathbf{E} = -[\mathbf{u}\times\mathbf{B} + \alpha\mathbf{B} - \beta\,\mathrm{curl}\,\mathbf{B}] \;. \tag{18}$$

For a thin disc with $\mathbf{u}{=}r\omega\mathbf{e}_\varphi$ and $\partial/\partial\varphi \equiv 0$ this becomes

$$E_r \;=\; \alpha B_r + \beta B'_\varphi \tag{19}$$

$$E_\varphi \;=\; \alpha B_\varphi - \beta B'_r \;. \tag{20}$$

Hence, it is evident that the requirement $[\![\mathbf{E}_{\mathrm{tang}}]\!] = \mathbf{0}$ leads just to the jump conditions (9), (10).

3 Discontinuities on arbitrary surfaces

Sokoloff (1997) derived a generalisation of the additional source term for the case of general surfaces of discontinuity by integrating the induction equation

$$\frac{\partial\mathbf{B}}{\partial t} = \mathrm{curl}\;[\mathbf{u}\times\mathbf{B} + \alpha\mathbf{B} - \beta\,\mathrm{curl}\,\mathbf{B}] \tag{21}$$

over the whole space. Let α be continuous on $\mathrm{I\!R}^3\backslash\mathcal{S}$ with $\mathcal{S}$ denoting an arbitrary, measurable surface. Let α_+ (α_-) be the value of $\alpha(\mathbf{x})$ on the positive (negative) side $\mathcal{S}_+$ ($\mathcal{S}_-$) of $\mathcal{S}$; let $[\![\alpha]\!] {:=} \alpha_+ {-} \alpha_-$ on $\mathcal{S}$. We assume that α and $\mathbf{u}$ are localised in space, B is square integrable, and that β grows not faster than $\mathcal{O}(\sqrt{r})$ for $r{\to}\infty$. Then, integration of Equation (21) over the whole space except an ε-neighbourhood $U_\varepsilon(\mathcal{S})$ of $\mathcal{S}$ gives

$$\frac{d}{dt}\int\limits_{\mathrm{I\!R}^3\backslash U_\varepsilon(\mathcal{S})} \mathbf{B}\,dx^3 = -\int\limits_{\partial U_\varepsilon(\mathcal{S})} [\mathbf{u}\times\mathbf{B} + \alpha\mathbf{B} - \beta\,\mathrm{curl}\,\mathbf{B}]\times d\mathbf{S} \;. \tag{22}$$

[1] $\mathbf{E}_{\mathrm{normal}}$, on the other hand, *does* jump, because discontinuities in the induction effects lead to surface charges.

The integral on the left-hand side should be regarded as a principial-value integral, cf. Dobler (1997). For $\varepsilon \to 0$ we get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} \mathbf{B}\, dx^3 \quad = \quad -\int_{\mathcal{S}} \Big([\mathbf{u}\times\mathbf{B} + \alpha\mathbf{B} - \beta \operatorname{curl} \mathbf{B}]_+ \\ -[\mathbf{u}\times\mathbf{B} + \alpha\mathbf{B} - \beta \operatorname{curl} \mathbf{B}]_- \Big) \times d\mathbf{S} \qquad (23) \end{aligned}$$

and the right hand side is what gave rise to Sokoloff's additional term.

According to (18), the expression in square brackets is simply $-\mathbf{E}$ and we get

$$\frac{d}{dt} \int_{\mathbb{R}^3} \mathbf{B}\, dx^3 = -\int_{\mathcal{S}} [\![\mathbf{E}]\!] \times d\mathbf{S} = \mathbf{0} \qquad (24)$$

because of (16). This result also follows directly from (15) and states that the magnetic moment $\vec{\mu} := \mathrm{v.\,p.} \int \mathbf{B}\, dx^3$ (cf. Sokoloff, 1997; Dobler, 1997) is conserved even when the coefficients in the induction equation (21) have surfaces of discontinuity.

Equation (24) clearly shows that Sokoloff's additional term is identically zero when appropriate account is taken of the (discontinuous) term $\beta \operatorname{curl} \mathbf{B}$.

4 The $\alpha\omega$-approximation for discontinuous α-effect

It seems that the neglection of the α-effect in the azimuthal equation (6) can not be justified when α' is a δ-function, as indicated in the Introduction and also at the end of the paper by Ruzmaikin *et al* (1979), and that this is a possible source of discrepancies with calculations for full $\alpha^2\omega$-dynamos with weak α-effect. However, this explanation can be ruled out. To show this, let us write down Equation (6) for Parker's model in dimensionless form

$$\frac{\partial B_\varphi}{\partial \tilde{t}} = R_\alpha (\tilde{\alpha} B_r)' + R_\omega B_r + (\tilde{\beta} B_\varphi')' \,, \qquad (25)$$

where dimensionless quantities are indicated by tilde and the prime now denotes differentiation with respect to $\tilde{z}$. The quantities $R_\alpha = \alpha_* h / \beta(0)$ and $R_\omega = G h^2 / \beta(0)$ are magnetic Reynolds numbers with α_* denoting a characteristic value of α (e. g. $\alpha_* = 1/(2h) \cdot \int_{-h}^{h} |\alpha|\, dz$), and in Parker's model we have $\tilde{\alpha}(\tilde{z}) = \operatorname{sgn} \tilde{z}$. For simplicity, we suppose $\tilde{\beta}(\tilde{z})$ to be continuous. The change to non-dimensional variables is a *formal* transformation and Equation (25) holds independent of whether $\alpha'(z)$ — or even $\alpha(z)$ — is bounded or not.

It is evident that for $\tilde{z}{\neq}0$ the first term on the right hand side can safely be neglected if $R_\omega/R_\alpha{\gg}1$. At $z{=}0$, the jump condition (10) and the equation for the time evolution of $B_\varphi(0)$ govern the behaviour of B_φ. They can be obtained by integrating (25) from $-\varepsilon$ to ε and retaining terms up to the order ε. For even modes we get

$$R_\alpha[\![\tilde{\alpha}]\!]B_r(0) + [\![B'_\varphi]\!] \;=\; 0 \tag{26}$$

$$\frac{d\tilde{B}_\varphi(0)}{dt} \;=\; R_\alpha\Big[\tilde{\alpha}(+0)B'_r(+0) + \tilde{\alpha}'(+0)B_r(0)\Big] + \\ + R_\omega B_r(0) + (\tilde{\beta}B'_\varphi)'(+0)\;. \tag{27}$$

Equations (26) and (27) give a good criterion to compare the influence of differential rotation and α-effect. In Equation (27), the α-effect can be neglected if $|R_\omega|{\gg}|R_\alpha|$, just as in Equation (25) for $\tilde{z}{\neq}0$.

In the jump condition (26), the first term is negligible (and thus B'_φ can be regarded as continuous) provided that

$$\left|R_\alpha\frac{B_r}{B_\varphi}\right| \ll 1\;. \tag{28}$$

The ratio of azimuthal to radial field can be estimated by

$$\left|\frac{B_\varphi}{B_r}\right| \sim |R_\omega|^m{\cdot}|R_\alpha|^n\;. \tag{29}$$

For the exponents m,n the values $m{=}n{=}1/2$ (which implies $B_\varphi/B_r{\sim}\sqrt{D}$; $D{=}R_\alpha R_\omega$ denotes the dynamo number) are given by Ruzmaikin *et al* (1988) for continuous profiles of α. Parker's results yield $m{=}1/3$, $n{=}{-}2/3$ for his model — cf. his equations (63), (64) or Section 5 of the present paper. Condition (28) then becomes

$$\frac{|R_\omega|^m}{|R_\alpha|^{1-n}} \gg 1\;. \tag{30}$$

Thus, for $m{=}n{=}1/2$ we get the condition $|R_\omega|{\gg}|R_\alpha|$ for the applicability of the $\alpha\omega$-approximation; for $m{=}1/3$, $n{=}{-}2/3$ we get $|R_\omega|{\gg}|R_\alpha|^5$. This latter condition has also been derived by Ruzmaikin *et al* (1980b). We note, however, that the exponents m, n for Parker's model are only valid for small growth rates γ. In particular, for $R_\alpha{=}\mathcal{O}(1)$, $|R_\omega|{\to}\infty$, the $\alpha\omega$-approximation may be invalid (as some preliminary numerical experiments seem to indicate), just as in the case of Moffatt's model (1) that will be discussed at the end of Section 6.

For odd modes, instead of (26), (27) we get simply

$$\frac{dB_\varphi(0)}{dt} = 0\ . \tag{31}$$

This confirms the finding of Ruzmaikin *et al* (1980b) that dipole modes are not sensitive to a jump in α at z=0 at all.

5 Analytical vs. numerical results

Once we have ruled out the possibility that discontinuous induction effects give rise to additional source terms, and having shown the applicability of the $\alpha\omega$-approximation, how can we explain the different properties of Parker's (1971) analytical solution compared to the numerical results by Ruzmaikin *et al* (1979) and others? The answer is that there are no severe discrepancies in the field structure, but the unlucky scaling of Parker's figure 5 led to its misinterpretation.

Ruzmaikin *et al* conclude from Equations (5), (6) that for G<0 (which is the case for Keplerian and galactic discs) the azimuthal field B_φ and the integral of the radial field $\int_0^h B_r\,dz$ must have opposite sign, while in Parker's figure B_x (corresponding to B_r) and B_y (i. e. B_φ) are both positive in the whole disc. But the latter is not true: In Parker's figure, B_x is scaled with a factor $R_\omega^{1/3}/R_\alpha^{2/3}$ relative to B_y. For G<0 this scaling factor is *negative*, thus the physical field component $B_x(z)$ is negative although its dimensionless counterpart plotted by Parker is positive. Taken into account this scaling, B_x has the appropriate sign and the qualitative behaviour of B_x and B_y is quite similar to that obtained from other distributions $\alpha(z)$.

Here we give the *exact result* for the stationary solution ($\partial\mathbf{B}/\partial t = \mathbf{0}$) of Equations (5), (6) with α-effect (2) and boundary conditions (7), (8) in the $\alpha\omega$-approximation:

$$B_r = C_1 e^{-\kappa|z|} + e^{\frac{\kappa}{2}|z|}\left(C_2\cos\frac{\sqrt{3}\kappa}{2}z + C_3\sin\frac{\sqrt{3}\kappa}{2}|z|\right) \tag{32}$$

$$\begin{aligned} B_\varphi = \frac{\kappa}{R_\alpha}\Bigg[& -C_1 e^{-\kappa|z|} + \\ & + e^{\frac{\kappa}{2}|z|}\left(\frac{C_2+\sqrt{3}C_3}{2}\cos\frac{\sqrt{3}\kappa}{2}z - \frac{\sqrt{3}C_2-C_3}{2}\sin\frac{\sqrt{3}\kappa}{2}|z|\right)\Bigg]\ . \end{aligned} \tag{33}$$

We have set β≡1, h=1; the parameter $\kappa := D^{1/3}$ is a solution of the equation

$$e^{\frac{3}{2}\kappa} + 2\cos\frac{\sqrt{3}\kappa}{2} = 0 \tag{34}$$

Table 1: Translation of some important symbols from Parker's notation to ours

Parker:	νQ_{123}	η	kh	$\eta k^2/G$	k_x	$\eta k k_x/G$
present paper:	$-\alpha$	β	$-\kappa = -D^{1/3}$	$R_\alpha^{2/3}/R_\omega^{1/3}$	k	$-khR_\alpha^{1/3}/R_\omega^{2/3}$

Table 2: Parameters for the first four exact solutions of Parker's $\alpha\omega$-dynamo model. For comparison, the critical dynamo numbers D_{Parker} according to Parker are given in the last line

$\kappa = D^{1/3}$	:	-1.84981	-5.44123	-9.06900	-12.69660
D	:	-6.32970	-161.098	-745.895	-2046.736
C_1	:	0.032850	$-1.4263 \cdot 10^{-4}$	$6.1808 \cdot 10^{-7}$	$-2.6784 \cdot 10^{-9}$
C_2	:	0.92836	0.86574	0.86603	0.86603
C_3	:	0.49806	0.50000	0.50000	0.50000
$D_{\mathrm{Parker}}^{1/3}$	:	-1.815	-5.43		

and the coefficients C_1, C_2, C_3 are determined by

$$C_1 = c(s+\sqrt{3}c) \tag{35}$$

$$C_2 = \frac{\sqrt{3}}{2}+2cs \tag{36}$$

$$C_3 = \frac{1}{2} - 2c^2 \tag{37}$$

with $c := \cos(\sqrt{3}\kappa/2)$, $s := \sin(\sqrt{3}\kappa/2)$. Note that our coefficients $C_{1,2,3}$ have nothing to do with Parker's $C_1 \dots C_4$.

For convenience, Table 1 lists the most important symbols in Parker's notation and in the one used here. Table 2 lists the first three critical dynamo numbers D as determined from Equation (34), together with the corresponding values of C_1, C_2, C_3.

Comparison of these results with Parker's formulas (62), (63) shows quite good agreement despite some approximations made by Parker at this point. In particular, he also found non-oscillating modes with very similar critical dynamo numbers. If one allows for slow radial variation of the magnetic field with a radial wave number $k \ll 1/h$, the z-component of the magnetic field can be calculated from $f(r) \cdot B_r = -B_z'/k$, where Parker gets $f(r) = \tan(kr - \pi/6)$, while a model of an infinite slab with exact treatment of the radial dependence yields $f(r) = J_0(kr)/J_1(kr)$ (cf. Dobler & Walker, 1997). This gives

$$B_z = f(r) \cdot \frac{k}{\kappa} \operatorname{sgn} z \Big[C_1 e^{-\kappa|z|} -$$

$$- e^{\frac{\kappa}{2}|z|} \left(C_1 \cos \frac{\sqrt{3}\kappa}{2} z + \frac{\sqrt{3}C_2 + C_3}{2} \sin \frac{\sqrt{3}\kappa}{2} |z| \right) \Bigg] . \quad (38)$$

Parker's z-component (65) looks considerably different for the first even mode, but for $kh \ll 1$ the vertical field component is small and irrelevant for the overall field structure.

In the limit of large negative dynamo numbers $\kappa \to -\infty$, the above equations reduce to

$$B_r = e^{\frac{\kappa|z|}{2}} \cos\Big(\frac{\sqrt{3}\kappa}{2}|z| - \frac{\pi}{6} \Big) \quad (39)$$

$$B_\varphi = \frac{\kappa}{R_\alpha} e^{\frac{\kappa|z|}{2}} \cos\Big(\frac{\sqrt{3}\kappa}{2}|z| + \frac{\pi}{6} \Big) \quad (40)$$

$$B_z = -f(r) \frac{k}{\kappa} e^{\frac{\kappa|z|}{2}} \sin \frac{\sqrt{3}\kappa}{2} z . \quad (41)$$

The same asymptotics follows from Parker's equations. As the exponential function decays very rapidly for $-\kappa \gg 1$, the field takes the form of a boundary layer located at the midplane $z=0$. This boundary layer can behave qualitatively very different for a smooth α-profile, where $|\alpha| \ll \alpha_*$ for $z \approx 0$. In fact, the asymptotic behaviour for $|D| \to \infty$ is the only point where the continuity properties of $\alpha(z)$ play an important rôle — but not a more prominent one than other details of $\alpha(z)$ near the midplane (cf. Ruzmaikin *et al*, 1988).

In order to verify our conclusion from Section 2 that there is no additional source term for discontinuous α-coefficients, we compared our exact result (32), (33), (38) with numerical solutions for the same problem. We applied a simple central-difference scheme with vertical grid size $\delta z = 2h/N$ as described by Baryshnikova & Shukurov (1987) and solved the resulting matrix eigenvalue problem by LAPACK routines. Two different approaches, one with step-like α-effect, $\alpha(z) = \operatorname{sgn} z$, the other with a resolved step, $\alpha(z) = \tanh(z/2\delta z)$, converged towards the same solution as we increased N. Figure 1 shows that the numerical results and our exact solution are in excellent agreement. Parker's solution — his equations (64)–(66) — which is obtained after some steps of approximation, is quite close to the exact result, apart from the (unimportant) z-component.

Next, we compared Parker's growth rates (the formula before his equation (61)) that were criticised by Ruzmaikin *et al* (1979), with our numerical solution. The result, shown in Figure 2, shows that the criticism is fully justified: there is no resemblance between the two results — except

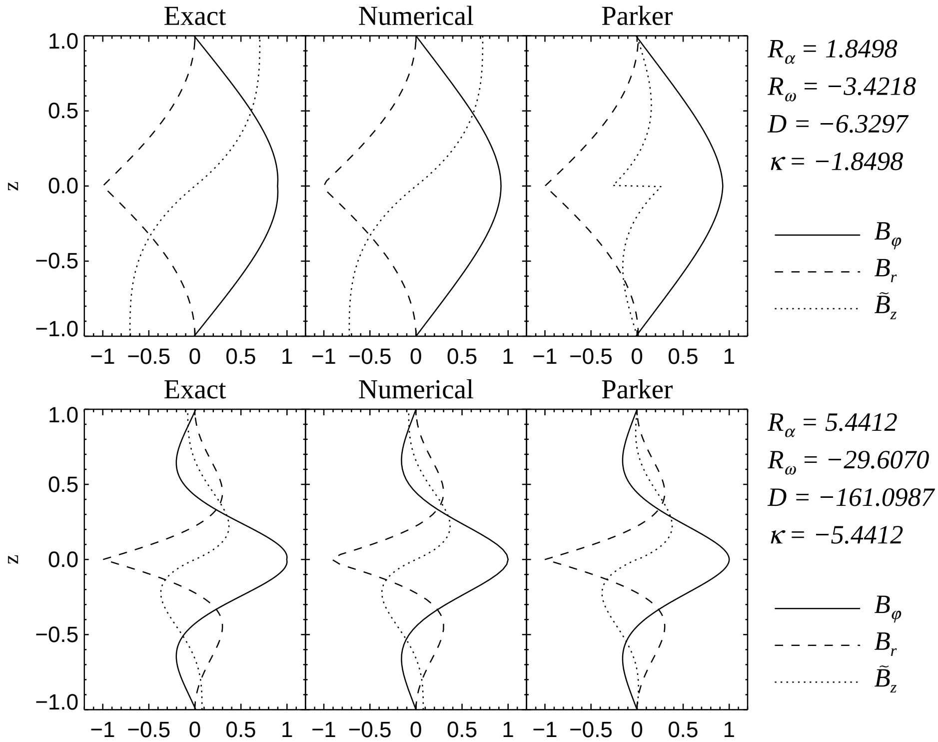

Figure 1: First two even modes of Parker's (1971) $\alpha\omega$-dynamo model. Left: exact result; middle: numerical solution with grid size $N{=}60$; right: Parker's equations (64)–(66). Shown are the azimuthal field B_φ (solid line), radial field B_r (dashed) and scaled vertical field $\tilde{B}_z{:=}B_z{\cdot}2R_\alpha/(kf(r))$ (dotted).

for the zeros of $\gamma(D)$. These coincide with the exact values, because the denominator in Parker's formula for $\gamma(D)$ is proportional to our Equation (34). However, the rest of the time-dependence has been lost due to the approximations made. This finding also overrides Moffatt's (1978) conclusion about the sensitivity of the growth rates to details of the dynamo model: Moffatt's growth rates are quite reasonable and must not be compared to Parker's, which are artefacts.

We emphasise that the deviations in the growth rates (Figure 2) are *not* due to the discontinuity of $\alpha(\mathrm{x})$ in Equation (2). There is no reason to believe that for $\gamma{\neq}0$ any noticeable differences between the exact solution of Parker's model and numerical solutions of the smoothed model (3) should occur. The grossly wrong behaviour of $\gamma(D)$ must be exclusively due to some additional approximations made by Parker, and an exact solution of the time-dependent dynamo model would reproduce the curves $\gamma(D)$ that we obtained numerically in Figure 2.

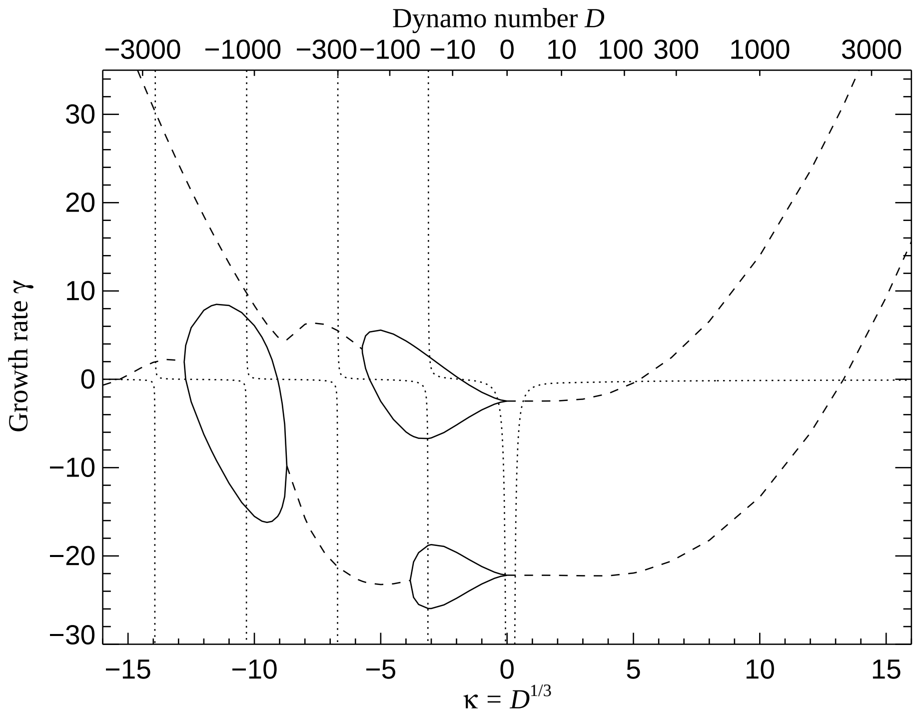

Figure 2: Numerically obtained growth rate γ for the first four even modes as a function of κ. Solid line: real growth rate; dashed line: real part of γ for oscillating modes. The dotted line represents Parker's result; his growth rates are real anywhere in the plotted range. Note that his zeros are exact despite the qualitatively wrong shape of the graph.

6 Conclusion

Our conclusion from the comparison of our exact solution (32), (33), (38) with numerical results is clear: models with discontinuous α-coefficient $\alpha(z) = \operatorname{sgn} z$ are fully compatible with numerical calculations and can be arbitrarily good approximated by continuous profiles $\alpha(z) = \operatorname{sgn}_\varepsilon z$ with $\lim_{\varepsilon\to 0} \operatorname{sgn}_\varepsilon z = \operatorname{sgn} z$. Although there is a term containing the derivative $\alpha'(z) \sim \delta(z)$ in the equations, nothing unexpected happens and in particular no additional terms appear in the induction equation.

When the induction coefficients themselves contain δ-functions like in Moffatt's (1978) model, things are a bit more difficult. In the course of solving Equations (5) and (6), products of generalised functions of type $\delta(z)\cdot\theta(z)$ appear, which are not mathematically well-defined as was stressed by Ruzmaikin *et al* (1980b). This can be illustrated as follows. From the identity

$$[\theta(x)]^n = \theta(x) \tag{42}$$

that holds at least in all points except $x=0$, we get by differentiating

$$n[\theta(x)]^{n-1}\delta(x) = \delta(x) \tag{43}$$

which implies $n\cdot[\theta(0)]^{n-1} = 1$. For different $n\in\mathbb{N}$ this leads to contradicting values of $\theta(0)$, which indicates that $\theta(x)\cdot\delta(x)$ is not well-defined. Other problems arise, when the θ-function and the δ-function are not "concentric" (cf. Ruzmaikin *et al*, 1980b), but this is unimportant here, since the θ-function is obtained directly from the δ-function, which ensures concentricity.

It would be far too restrictive to exclude any product of generalised functions from consideration. Rather, working with approximations θ_ε, δ_ε of the Heaviside and Dirac function, one gets a feeling of what is allowed in this field. The apparent contradiction in Equation (43) is connected with the fact that there is evidently no continuous θ_ε, for which Equation (42) holds for all $n\in\mathbb{N}$. Thus, one has to avoid applying Equation (42) or the more general form $f(\theta(x)) = f(0) + \theta(x)\cdot[f(1)-f(0)]$ and work with the original expression $[\theta(x)]^n$ or $f(\theta(x))$. Results achieved in this way will be free of the mathematical problems mentioned above. If necessary, one could instead exploit structural similarities of special approximations, as for example $\operatorname{sgn}_\varepsilon(x) = \operatorname{sgn} x\cdot \exp(-\varepsilon/x^2)$ with $\operatorname{sgn}_\varepsilon^3 = \operatorname{sgn}_{3\varepsilon}$ *etc.*

Analog relations to (26), (27) can be determined for Moffatt's (1978) type of model (see Equation (1)) in order to determine the applicability condition for the $\alpha\omega$-approximation. As in Section 4, the equation for $dB_\varphi(0)/dt$ yields no additional information, while the "jump condition" now reads

$$R_\alpha \overline{B_r}(\tilde{\zeta}) \cdot \int\limits_{\tilde{\zeta}-0}^{\tilde{\zeta}+0} \tilde{\alpha}(\tilde{z})\, d\tilde{z} + [\![B_\varphi]\!] = 0 \tag{44}$$

with $\overline{B_r}(\tilde{\zeta}) := (B_r(\tilde{\zeta}+0) + B_r(\tilde{\zeta}-0))/2$. Thus, we again can neglect the term $(\alpha B_r)'$ in Equation (6) (and regard B_φ as continuous) provided that

$$\left| R_\alpha \frac{B_r}{B_\varphi} \right| \ll 1\ . \tag{45}$$

The asymptotic ratio B_r/B_φ for $|R_\omega|\gg|R_\alpha|$ has been given by Ruzmaikin *et al* (1980b). For $D\approx D_{\text{crit}}$ their result is $B_r/B_\varphi \sim (|R_\alpha|/|R_\omega|)^{1/2}$, which gives $|R_\omega|\gg|R_\alpha|^3$ as applicability condition for the $\alpha\omega$-approximation. This is trivially fulfilled because we have $R_\alpha\sim D_{\text{crit}}/R_\omega$ in this case.

If $R_\alpha=\mathcal{O}(1)$ is fixed, the growth rate increases with R_ω and we have $B_r/B_\varphi \sim R_\omega \ln|D|$ for $R_\omega\to\infty$. Now our condition would read $D\ln|D|\ll 1$, which obviously contradicts our assumption $R_\omega \gg R_\alpha = \mathcal{O}(1)$. Hence, we find that for $R_\alpha = \mathcal{O}(1)$ the $\alpha\omega$-approximation is not permissible in Moffatt's model, independent of the magnitude of R_ω. This was also concluded by Ruzmaikin *et al* (1980b), but based on the more naïve assumption that the $\alpha\omega$-approximation holds whenever $B_r/B_\varphi \ll 1$ instead of our condition (45).

Acknowledgements

I am grateful to Anvar Shukurov for helpful remarks and notes. In addition, I want to thank Dmitrij Sokoloff for initiating the present paper and for stimulating and controversial discussions.

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